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Analytical Study of Blocking Sound Transmission through a Wall of Passively Oscillating Panels

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Abstract

Existing work has shown that sound transmission through a partition composed of two panels in a duct may be reduced at frequencies between the two panels' natural frequencies as the panels passively oscillate out of phase. The significance of this sound reduction method known as Alternate Resonance Tuning (ART) is that sound transmission through a barrier can be significantly reduced by tuning the panels to different natural frequencies without increasing the mass or the damping of the barrier. ART has been studied for the case of two panels in a duct subject to sound at normal incidence. Through analytical methods, this thesis extends ART to an infinite array of panels subject to sound at oblique incidence. As a result, the theoretical studies of ART more accurately model actual physical sound-blocking structures which are subject to sound at oblique angles. First I explore ART in its original framework: two panels in a duct. This requires an understanding of the motion of one panel in a duct, hence this simplified case is the starting point for the analysis. The system is then expanded to an infinitely tall array of panels in which every other panel is identical. The acoustic intensity radiated from this acoustic barrier is calculated, and it is determined that in order to minimize sound transmission, the panels should have low fluid loading, large structural damping, large panel height, and very oblique incidence angles.

Contents

1 Introduction

Sound-blocking structures such as the cabin walls of an aircraft or a shared partition between two apartments contribute to the peace and quiet of everyday life. In the applied field of architectural acoustics, reducing sound transmission is achieved by a number of methods including increasing the mass of the materials, adding fiberglass insulation, or introducing an air space between two walls. Some of these solutions are not cost-effective, however, and in cases such as the cabin of an airplane, there are constraints to minimize the total mass of the acoustic barrier.

Methods have been developed to block sound transmission when the acoustic barrier even has low structural damping. Alternate Resonance Tuning (ART) is a noise reduction technique in which two panels are tuned to different natural frequencies. At frequencies between the two resonance frequencies, the panels move out of phase and the radiated acoustic power is significantly reduced[1−5]. Experimental work has verified this for normal incidence of panels mounted on a compliant frame in a duct[6]. Also, Analytical studies have been conducted on the oblique incidence of an acoustic barrier composed of identical membranes[7]. This thesis will use these ideas to investigate ART at oblique incidence.

The goal of this project is to develop an analysis for determining the sound transmission through the structural barrier shown in Figure 1 which is composed of many panels with varying dynamic properties. Each panel is treated as a mass-spring-damper that oscillates in response to the acoustic forcing. Then, methods of reducing sound transmission are investigated by adjusting certain physical parameters. To start this process, it is beneficial to understand ART at normal incidence by first considering just two panels in a duct and observing the outcome when these panels oscillate out of phase. To understand the underlying physics of the two panels' interacting motion and verify the concepts analytically, it is helpful to think about these two panels within a framework that is already understood, namely the case of one panel in a duct.

After analyzing the panels in the duct, the calculation will address the structure in Figure 1. The depth of the barrier in the $\hat{z}$ direction will be neglected for simplicity, and the analysis will treat the barrier as an infinitely tall stack of panels that oscillate in the $\hat{x}$ direction. For both this case and that of two panels in a duct, the power transmitted through the barrier is calculated as a function of panel dynamics such as structural damping, natural frequency, panel height, etc. The goal is to observe how these parameters may be adjusted to minimize the transmitted power.

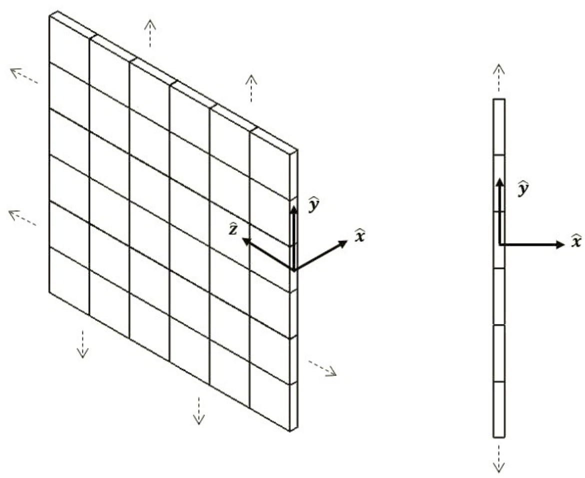

Figure 1: A sound-blocking structure is deconstructed into infinitely many panels that lie in the y-z plane. Working in two dimensions, the drawing on the right illustrates the viewpoint that is of primary focus: panels oscillating left and right in the $\hat{x}$ direction.

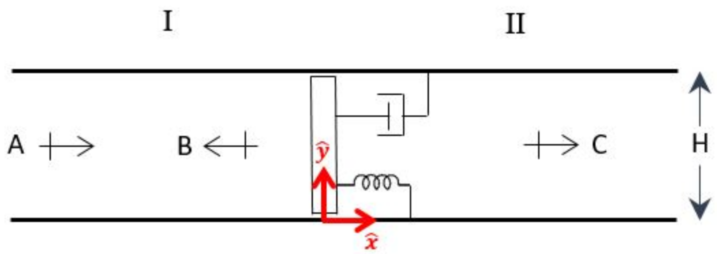

Figure 2: A single panel (modeled as a mass-spring-damper) lies in a duct of height H. A plane wave of amplitude A strikes the panel at normal incidence and creates reflected and transmitted plane waves with amplitudes B and C, respectively.

2 Panels in a Duct

2.1 One Panel, One-Dimensional Case

The analysis begins with one panel in a duct of height H is shown in Figure 2. A pressure wave with known amplitude **A** strikes the panel at normal incidence. A panel can be modeled as a mass-spring-damper with mass m, mechanical resistance R, spring constant s_c, and natural frequency ω_N. When the mass oscillates in response to plane wave **A**, reflected and transmitted plane waves with amplitudes **B** and **C**, respectively, are created. The current objective is to determine expressions for amplitudes **B** and **C** given that the sound source (amplitude of incident pressure wave, **A**) is known. The origin is placed at the bottom of the duct at the location of the panel (throughout this paper, the displacement of the panel is considered small enough that the panel is always located at $x = 0$). The region of the duct for which $x < 0$ will be referred to as Region I and Region II is where $x > 0$. This system is one-dimensional because the pressures and velocities have no y dependence and only vary in the $\hat{x}$ direction. The incoming, reflected, and transmitted sound waves have plane wave fronts that lie in the y-z plane. Note that throughout this paper, the system will be considered linear such that the incident, reflected, and transmitted pressures (and velocities) are all at the same angular frequency, ω. Complex expressions are bolded.

Expressions for the pressure in Regions I and II (Eqns 2 and 3) can be found by solving the one-dimensional wave equation, Eqn 1[8]. c is the speed of sound, t is time, ω is the angular frequency, and k is wave number.

$$\vec{\nabla}^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \longrightarrow \frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{1}$$

$$\mathbf{p}_{\mathrm{I}}(x,t) = \mathbf{A}e^{i(\omega t - kx)} + \mathbf{B}e^{i(\omega t + kx)} \tag{2}$$

$$\mathbf{p}_{\mathrm{II}}(x,t) = \mathbf{C}e^{i(\omega t - kx)} \tag{3}$$

Wave velocity **u** (Eqns 5 and 6) can be found from the pressure through the one-dimensional linear Euler equation Eqn 4, also known as the momentum equation, where ρ_0 is the constant equilibrium density of the fluid[8].

$$\rho_0 \frac{\partial u}{\partial t} = -\vec{\nabla} p \longrightarrow \rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} \tag{4}$$

$$\mathbf{u}_{\text{I}}(x,t) = \frac{\mathbf{A}}{\rho_0 c} e^{i(\omega t - kx)} - \frac{\mathbf{B}}{\rho_0 c} e^{i(\omega t + kx)} \tag{5}$$

$$\mathbf{u}_{\text{II}}(x,t) = \frac{\mathbf{C}}{\rho_0 c} e^{i(\omega t - kx)} \tag{6}$$

At the panel ($x = 0$), $\mathbf{u}_{\text{I}} = \mathbf{u}_{\text{II}}$. As a result:

$$\left(\frac{\mathbf{A}}{\rho_0 c} - \frac{\mathbf{B}}{\rho_0 c}\right) e^{i(\omega t - kx)} = \frac{\mathbf{C}}{\rho_0 c} e^{i(\omega t - kx)} \tag{7}$$

Consequently,

$$\mathbf{C} = \mathbf{A} - \mathbf{B} \tag{8}$$

The incident pressure coefficient $\mathbf{A}$ is given, but the reflected and transmitted coefficients, $\mathbf{B}$ and $\mathbf{C}$ respectively, must be found in order to complete the expressions for pressure and velocity given above. $\mathbf{A}$ and $\mathbf{B}$ are related through the complex mechanical impedance, $\mathbf{Z_m}$, which is the ratio between the net force on the panel ($F_{\text{I}} - F_{\text{II}}$) and its velocity[8]. Force is Pressure × Area, but since only the x-y plane is being considered, the length of the panel in the z-dimension is ignored and the panel height H is considered the "area". From this point onward, the $e^{i\omega t}$ term is omitted because it is present in all pressure and velocity terms and subsequently cancels out of final expressions.

$$\mathbf{Z_m} = \left.\frac{\mathbf{F}_{\text{I}} - \mathbf{F}_{\text{II}}}{\mathbf{u}_{\text{I}}}\right|_{x=0} = \left.\frac{H(\mathbf{p}_{\text{I}} - \mathbf{p}_{\text{II}})}{\mathbf{u}_{\text{I}}}\right|_{x=0} = \frac{H(\mathbf{A} + \mathbf{B} - \mathbf{C})}{\frac{1}{\rho_0 c}(\mathbf{A} - \mathbf{B})} = \frac{2H\mathbf{B}}{\frac{1}{\rho_0 c}(\mathbf{A} - \mathbf{B})} \tag{9}$$

Rearranging Eqn 9 yields Eqn 10, an expression for $\mathbf{B}$ in terms of $\mathbf{A}$ which is given, assuming the sound source is known.

$$\mathbf{B} = \mathbf{A}\left(\frac{\mathbf{Z_m}}{\mathbf{Z_m} + 2H\rho_0 c}\right) \tag{10}$$

In order to have a complete expression for $\mathbf{B}$, an expression for $\mathbf{Z_m}$ in terms of the panel dynamics is needed. From Newton's second law, the force balance on the panel (mass m, mechanical resistance R, spring constant s_c, and displacement x) is given in Eqn 11.

$$m\ddot{x} = -R\dot{x} - s_c x + F_{\text{I}} - F_{\text{II}} \tag{11}$$

After some manipulation (intermediate steps given in the Appendix), the impedance is written in terms of the panel dynamics in Eqn 12.

$$\mathbf{Z_m} = R + i\left(m\omega - \frac{s_c}{\omega}\right) \tag{12}$$

For convenience, the mechanical impedance and its constituent variables are re-expressed in dimensionless form, with the unitless impedance $\bar{\mathbf{Z}}$ written in terms of acoustic damping (also known as fluid loading) ζ_{ac}, structural damping ζ, and $\bar{\omega}$, the ratio of the angular frequency ω to the natural frequency of the panel ω_N.

$$\bar{\mathbf{Z}} = \frac{\mathbf{Z_m}}{\rho_0 c H} = \frac{1}{2\zeta_{ac}}\left[2\zeta + i\bar{\omega}\left(1 - \frac{1}{\bar{\omega}^2}\right)\right] \tag{13}$$

$$\zeta_{ac} = \frac{\rho_0 c H}{2\sqrt{s_c m}},\ \zeta = \frac{R}{2m\omega},\ \bar{\omega} = \frac{\omega}{\omega_N}$$

Now $\mathbf{B}$ can be re-expressed in terms of A and $\bar{\mathbf{Z}}$:

$$\mathbf{B} = \mathbf{A}\left(\frac{\bar{\mathbf{Z}}}{\bar{\mathbf{Z}}+2}\right) \tag{14}$$

Then, combining Eqns 8 and 14, $\mathbf{C}$ is also expressed in terms of A and $\bar{\mathbf{Z}}$:

$$\mathbf{C} = \mathbf{A}\left(\frac{2}{\bar{\mathbf{Z}}+2}\right) \tag{15}$$

At this point, complete expressions for the acoustic pressure and velocity in Regions *I* and *II* have been determined (now that the coefficients $\mathbf{B}$ and $\mathbf{C}$ are known in the expressions for pressure and velocity). Next, when a second panel is introduced to the duct, it will be helpful to verify that the solution reduces to Eqns 14 and 15 when the two panels are identical and behave as one panel, as was calculated in this section.

2.2 Two Panels, One-Dimensional Case

As shown in Figure 3, the duct of height H now has two panels, a and b, each of height $\frac{H}{2}$. When the duct height H is much smaller than the wavelength of the source sound, then the problem is one-dimensional and the pressure has no y dependence: pressure at $y > \frac{H}{2}$ is the same as at $y < \frac{H}{2}$, or $\mathbf{p}_a = \mathbf{p}_b$. The panel velocity (velocity at $x = 0$), however does have a piecewise y dependence.

$$\mathbf{u}\Big|_{x=0} = \begin{cases} \mathbf{u}_a & \frac{H}{2} > y > H \\ \mathbf{u}_b & 0 < y < \frac{H}{2} \end{cases}$$

If the panels have different dynamics (fluid loading ζ_{ac}, structural damping ζ_a, or natural frequency) then the panels will respond to the incident pressure differently and $\mathbf{u}_a \neq \mathbf{u}_b$.

Just as in the one panel one-dimensional case, the one-dimensional wave equation (Eqn 1) can be used to find expressions for the pressure, and the one-dimensional momentum equation (Eqn 4) can be used to find the velocities in Regions *I* and *II*.

$$\mathbf{p}_{\mathrm{I}}(x,t) = \mathbf{A}e^{i(\omega t - kx)} + \mathbf{B}e^{i(\omega t + kx)} \tag{16}$$

$$\mathbf{p}_{\mathrm{II}}(x,t) = \mathbf{C}e^{i(\omega t - kx)} \tag{17}$$

$$\mathbf{u}_{\mathrm{I}}(x,t) = \frac{\mathbf{A}}{\rho_0 c}e^{i(\omega t - kx)} - \frac{\mathbf{B}}{\rho_0 c}e^{i(\omega t + kx)} \tag{18}$$

$$\mathbf{u}_{\mathrm{II}}(x,t) = \frac{\mathbf{C}}{\rho_0 c}e^{i(\omega t - kx)} \tag{19}$$

The mass of the fluid is conserved such that the volumetric flow rate (velocity × area) through the duct is the same as that through the upper and lower portions of the duct. Eqns 20 and 21 reflect this for both Regions *I* and *II*:

$$\mathbf{u}_{\mathrm{I}}H = \mathbf{u}_a\left(\frac{H}{2}\right) + \mathbf{u}_b\left(\frac{H}{2}\right) \tag{20}$$

$$\mathbf{u}_{\mathrm{II}}H = \mathbf{u}_a\left(\frac{H}{2}\right) + \mathbf{u}_b\left(\frac{H}{2}\right) \tag{21}$$

Consequently, $\mathbf{u}_{\mathrm{I}} = \mathbf{u}_{\mathrm{II}}$, and from Eqns 18 and 19, it follows that $\mathbf{C} = \mathbf{A} - \mathbf{B}$. As before, expressions for the complex mechanical impedances (one for each panel) will be used to find $\mathbf{B}$ and $\mathbf{C}$ in terms of $\mathbf{A}$.

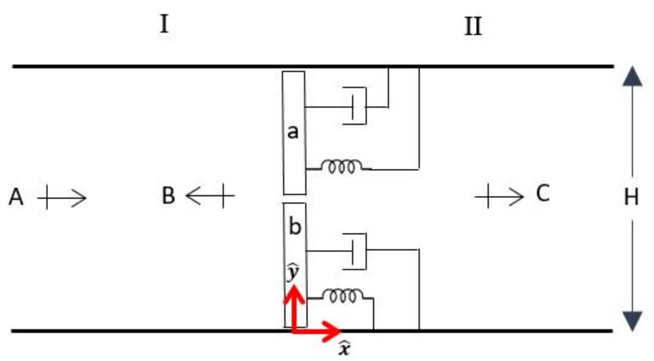

Figure 3: Two panels, labeled a and b, are in a duct of height H (panel height is now $\frac{H}{2}$). A plane wave of amplitude **A** strikes the panel at normal incidence and creates reflected and transmitted plane waves with amplitudes **B** and **C**, respectively. This figure depicts the one-dimensional case in which pressure has no y dependence. As a result, the incoming, reflected and transmitted waves are all plane fronts in the y-z plane.

$$\mathbf{Z}_{m_a} = \frac{\mathbf{F}_{\mathrm{I_a}} - \mathbf{F}_{\mathrm{II_a}}}{\mathbf{u}_a} = \frac{\frac{H}{2}(\mathbf{p}_{\mathrm{I}} - \mathbf{p}_{\mathrm{II}})}{\mathbf{u}_a} = \left(\frac{H}{2}\right)\left(\frac{\mathbf{A} + \mathbf{B} - \mathbf{C}}{\mathbf{u}_a}\right) = \frac{\mathbf{B}H}{\mathbf{u}_a} \tag{22}$$

Rearranging Eqn 22 to solve for $\mathbf{u}_a$ yields:

$$\mathbf{u}_a = \frac{\mathbf{B}H}{\mathbf{Z}_{m_a}} \tag{23}$$

Repeating this process for panel b gives two analogous equations.

$$\mathbf{Z}_{m_b} = \frac{\mathbf{F}_{\mathrm{I_b}} - \mathbf{F}_{\mathrm{II_b}}}{\mathbf{u}_b} = \frac{\frac{H}{2}(\mathbf{p}_{\mathrm{I}} - \mathbf{p}_{\mathrm{II}})}{\mathbf{u}_b} = \left(\frac{H}{2}\right)\left(\frac{\mathbf{A} + \mathbf{B} - \mathbf{C}}{\mathbf{u}_b}\right) = \frac{\mathbf{B}H}{\mathbf{u}_b} \tag{24}$$

$$\mathbf{u}_b = \frac{\mathbf{B}H}{\mathbf{Z}_{m_b}} \tag{25}$$

The expressions for $\mathbf{u}_a$ and $\mathbf{u}_b$ from Eqns 23 and 25 can be substituted into Eqn 20. The resulting expression can be rearranged, producing the following expression for **B**.

$$\mathbf{B} = \mathbf{A}\left(\frac{\mathbf{Z}_{m_a}\mathbf{Z}_{m_b}}{\frac{\rho_0 cH}{2}\mathbf{Z}_{m_a} + \frac{\rho_0 cH}{2}\mathbf{Z}_{m_b} + \mathbf{Z}_{m_a}\mathbf{Z}_{m_b}}\right) \tag{26}$$

After repeating the nondimensionalizing process used in the one panel one-dimensional case, final expressions for **B** and **C** and consequently, the acoustic pressure and velocity everywhere, are found:

$$\mathbf{B} = \mathbf{A}\left(\frac{\bar{\mathbf{Z}}_a\bar{\mathbf{Z}}_b}{\bar{\mathbf{Z}}_a + \bar{\mathbf{Z}}_a + \bar{\mathbf{Z}}_a\bar{\mathbf{Z}}_b}\right) \tag{27}$$

$$\mathbf{C} = \mathbf{A}\left(\frac{\bar{\mathbf{Z}}_a + \bar{\mathbf{Z}}_b}{\bar{\mathbf{Z}}_a + \bar{\mathbf{Z}}_a + \bar{\mathbf{Z}}_a\bar{\mathbf{Z}}_b}\right) \tag{28}$$

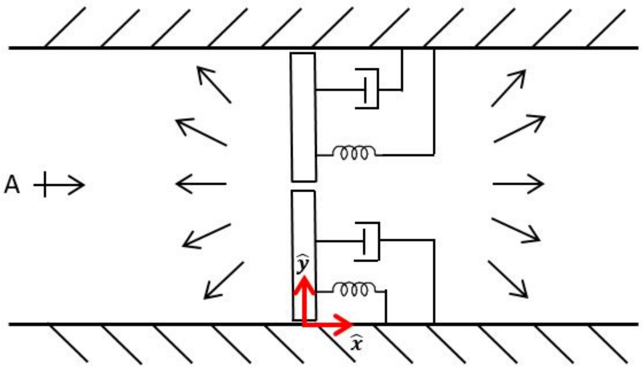

Figure 4: Now that pressure varies in the $\hat{y}$ direction, oblique waves radiate away from the panels due to the back-loading effect. The top and bottom of the duct are rigid: the y-component of the velocity is zero.

$$\bar{\mathbf{Z}}_a = \frac{\mathbf{Z_{m_a}}}{\rho_0 c \frac{H}{2}} = \frac{1}{\zeta_{aca}}\left[2\zeta_a + i\bar{\omega}\left(1 - \frac{1}{\bar{\omega}^2}\right)\right] \tag{29}$$

$$\bar{\mathbf{Z}}_b = \frac{\mathbf{Z_{m_b}}}{\rho_0 c \frac{H}{2}} = \frac{1}{\zeta_{acb}}\left[2\zeta_b + i\frac{\bar{\omega}}{\omega_R}\left(1 - \frac{\omega_R^2}{\bar{\omega}^2}\right)\right] \tag{30}$$

$$\zeta_{aca} = \frac{\rho_0 cH}{2\sqrt{s_{c_a} m_a}},\ \zeta_a = \frac{R}{2m_a\omega},\ \zeta_{acb} = \frac{\rho_0 cH}{2\sqrt{s_{c_b} m_b}},\ \zeta_b = \frac{R}{2m_b\omega},\ \bar{\omega} = \frac{\omega}{\omega_{N_a}},\ \omega_R = \frac{\omega_{N_b}}{\omega_{N_a}}$$

2.3 Two Panels, Two-Dimensional Case

From this point onward, complex terms will not be bolded: It can be assumed that all terms are complex except parameters such as H, ρ_0, c, ζ_{ac}, ζ, $\bar{\omega}$, and ω_R.

Now the same two panels in a duct will be treated in two dimensions: pressure and velocity vary in the $\hat{y}$ direction. There is a "sloshing" or back-loading effect created when the panels move out of phase from one another. This produces higher modes of pressure and velocity such that waves radiate at nonzero angles from the axis normal to the panels as shown in Figure 4.

Let the two-dimensional velocity vector be expressed as a sum of of horizontal and vertical components: $\vec{U} = u\hat{x} + v\hat{y}$. Recall that the momentum equation relates pressure and velocity vectors:

$$\rho_0 \frac{\partial \vec{U}}{\partial t} = -\vec{\nabla} p \tag{31}$$

Equating the $\hat{x}$ and $\hat{y}$ components of both sides independently gives the relations in Eqns 32 and 33.

$$i\omega\rho_0 u = -\frac{\partial p}{\partial x} \tag{32}$$

$$i\omega\rho_0 v = -\frac{\partial p}{\partial y} \tag{33}$$

The wave equation in two-dimensions, Eqn 34, is used to find an expression for pressure.

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{34}$$

First the constraints in the $\hat{y}$ direction are considered. The walls of the duct are rigid and create boundary conditions: the $\hat{y}$ component of velocity, v, is zero at $y = 0$ and $y = H$. According to Eqn 33, $\frac{\partial p}{\partial y}$ is zero at $y = 0$ and $y = H$ as well. These boundary conditions as well as the wave equation yield the expression for pressure in Eqn 35 where P_m is the modal pressure amplitude that will be defined shortly.

$$p(x, y, t) = \sum_{m=0}^{\infty} P_m \cos\left(\frac{m\pi y}{H}\right) e^{i(\omega t - k_{x_m} x)} \tag{35}$$

The momentum equation is now evaluated in the $\hat{x}$ direction (Eqn 32) using the expression for pressure in Eqn 35. The velocity of the fluid in the $\hat{x}$ direction at $x = 0$ is of interest because that gives the velocity of the barrier (the two panels). The $e^{i\omega t}$ term is dropped to reduce clutter as was previously done.

$$u\bigg|_{x=0} = \frac{i}{\rho_0 ck}\frac{\partial p}{\partial x}\bigg|_{x=0} = \sum_{m=0}^{\infty} P_m \frac{k_{x_m}}{\rho_0 ck} \cos\left(\frac{m\pi y}{H}\right) = \sum_{m=0}^{\infty} \hat{P}_m \cos\left(\frac{m\pi y}{H}\right) \tag{36}$$

$$\hat{P}_m = P_m \frac{k_{x_m}}{\rho_0 ck}$$

The velocity of the barrier, $U_s(y)$, and thus the fluid at $x = 0$ is now expressed as a sum of even functions. It follows then that $\hat{P}_m$ is the coefficient of the even terms of the Fourier series. But just as in the two panel one-dimensional case, the same velocity can be written as a piecewise expression:

$$u\bigg|_{x=0} = U_s(y) = \begin{cases} U_a & \frac{H}{2} > y > H \\ U_b & 0 < y < \frac{H}{2} \end{cases}$$

Eqn 36 is a Fourier series representation of this piecewise function. The immediate goal will be to find the Fourier series coefficients $\hat{P}_m$ after which P_m is easily found, thus completing the expression for pressure in Eqn 35. The calculations of the Fourier coefficients in Eqns 37 and 38 are shown in greater detail in the appendix. Note that there has been a shift in the index m to $2m - 1$.

$$\hat{P}_0 = \frac{U_a + U_b}{2} \tag{37}$$

$$\hat{P}_{m>0} = \frac{2}{(2m-1)\pi}(-1)^{m+1}(U_b - U_a) \tag{38}$$

Recall that $P_m = \hat{P}_m \frac{\rho_0 ck}{k_{x_m}}$. Substituting in these expressions to that of pressure from Eqn 35 yields:

$$p(x, y, t) = \frac{\rho_0 c}{2}(U_a + U_b) e^{i(\omega t - kx)} + \sum_{m=1}^{\infty} \frac{2k\rho_0 c}{(2m-1)\pi k_{x_m}}(-1)^{m+1}(U_b - U_a)\cos\left(\frac{(2m-1)\pi y}{H}\right) e^{i(\omega t - k_{x_m} x)} \tag{39}$$

k_{x_m} comes from the dispersion relation, $k_{x_m}^2 + k_{y_m}^2 = k^2$, which is derived from the wave equation. Here, the coefficient of y in the cosine term indicates that $k_{y_m} = \frac{(2m-1)\pi}{H}$. It follows that

$$k_{x_m} = \sqrt{k^2 - \left(\frac{(2m-1)\pi}{H}\right)^2} = -i\sqrt{\left(\frac{(2m-1)\pi}{H}\right)^2 - k^2} \tag{40}$$

Find Expressions for $\mathbf{U_a}$ and $\mathbf{U_b}$

Up until this point, the calculation was conducted assuming that U_a and U_b were known. It is now necessary to find expressions for U_a and U_b, and thereupon substutute them back into Eqn 39.

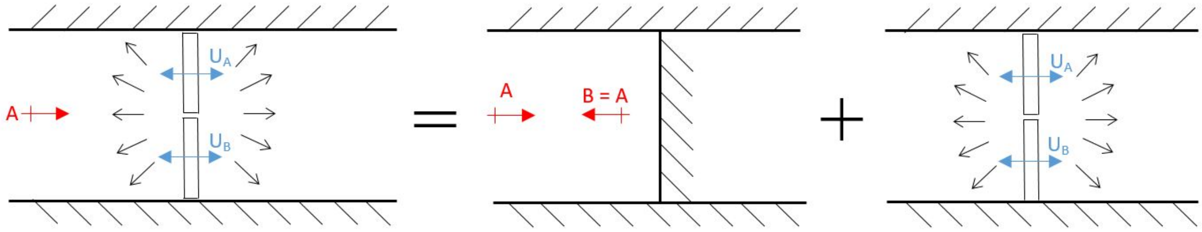

Figure 5: It is possible to decompose the factors contributing to the total force on the panels into two processes: The force due to the incoming pressure wave of amplitude A in which the panels do not move and there is perfect reflection (such that the reflected amplitude $B = A$), and the pressure created exclusively from the motion of the panels (no incoming pressure wave).

As previously shown, the mechanical impedance of a barrier is a transfer function between the force on the barrier and its velocity. In order to eventually find expressions for U_a and U_b then, the total force on each panel must be calculated. There are two contributions to the force on a panel: The incident pressure wave, A, from the sound source (the "blocked force") and the pressure at $x = 0$ due to the motion of the panels alone (the "backloading"). After finding the contribution of these forces independently, superposition will be used to calculate the total force on each panel, as shown in Figure 5.

To find the blocked force, there must be no motion of the barrier at all. The panels are then replaced with a rigid partition, creating perfect reflection. The pressure at the barrier (accounting for the incident and reflected waves) is thus twice that of the incident wave. Force is simply Pressure × Area, but since the depth of the duct in what would be the $\hat{z}$ direction is neglected, the panel height $H/2$ is the effective area. The blocked force on a panel is therefore

$$F_{\text{Blocked}_a} = F_{\text{Blocked}_b} = 2A\left(\frac{H}{2}\right) = AH \tag{41}$$

The force due to the panel motion is calculated by integrating the pressure (Eqn 39) along a panel. There is a -2 prefactor that accounts for the phenomena in which the pressure "pushes" from the right and "pulls" from the left, doubling the force in the $-\hat{x}$ direction. The backloading forces on panels a and b are given in Eqns 42 and 43, respectively.

$$\begin{aligned} F_{\text{Backloading}_a} &= -2\int_{\frac{H}{2}}^{H} p\,\text{dy} \\ &= -\int_{\frac{H}{2}}^{H} \rho_0 c\,(U_a + U_b)\,e^{i(\omega t - kx)} + \sum_{m=1}^{\infty} \frac{4k\rho_0 c}{(2m-1)\,\pi k_{x_m}} (-1)^{m+1} (U_b - U_a) \cos\left(\frac{(2m-1)\,\pi y}{H}\right) \text{dy} \\ &= -H\rho_0 c\left[\frac{U_a + U_b}{2} - \sum_{m=1}^{\infty} \frac{4k}{(2m-1)^2\,\pi^2 k_{x_m}} (U_b - U_a)\right] \end{aligned} \tag{42}$$

$$F_{\text{Backloading}_b} = -H\rho_0 c\left[\frac{U_a + U_b}{2} - \sum_{m=1}^{\infty} \frac{4k}{(2m-1)^2\,\pi^2 k_{x_m}} (U_a - U_b)\right] \tag{43}$$

Through the superposition of these effects, shown in Figure 5, $F_{\text{Total}} = F_{a/b} = F_{\text{Blocked}} + F_{\text{Backloading}}$. F_a and F_b are also related to the panel velocities and mechanical impedances.

$$F_a = AH - H\rho_0 c\left[\frac{U_a + U_b}{2} - \sum_{m=1}^{\infty}\frac{4k}{(2m-1)^2\pi^2 k_{x_m}}(U_b - U_a)\right] = Z_{m_a}U_a \tag{44}$$

$$F_b = AH - H\rho_0 c\left[\frac{U_a + U_b}{2} - \sum_{m=1}^{\infty}\frac{4k}{(2m-1)^2\pi^2 k_{x_m}}(U_a - U_b)\right] = Z_{m_b}U_b \tag{45}$$

Eqns 44 and 45 are a system of two equations and two unknowns: U_a and U_b. Solving for U_a and U_b yields

$$U_a = \frac{AH\left(Z_{m_b} + \sum_{m=0}^{\infty}\frac{8kH\rho_0 c}{(2m-1)^2\pi^2 k_{x_m}}\right)}{Z_{m_a}Z_{m_b} + (Z_{m_a} + Z_{m_b})\left(\frac{H\rho_0 c}{2} + \sum_{m=0}^{\infty}\frac{4kH\rho_0 c}{(2m-1)^2\pi^2 k_{x_m}}\right) + \sum_{m=0}^{\infty}\frac{8k(H\rho_0 c)^2}{(2m-1)^2\pi^2 k_{x_m}}} \tag{46}$$

$$U_b = \frac{AH\left(Z_{m_a} + \sum_{m=0}^{\infty}\frac{8kH\rho_0 c}{(2m-1)^2\pi^2 k_{x_m}}\right)}{Z_{m_a}Z_{m_b} + (Z_{m_a} + Z_{m_b})\left(\frac{H\rho_0 c}{2} + \sum_{m=0}^{\infty}\frac{4kH\rho_0 c}{(2m-1)^2\pi^2 k_{x_m}}\right) + \sum_{m=0}^{\infty}\frac{8k(H\rho_0 c)^2}{(2m-1)^2\pi^2 k_{x_m}}} \tag{47}$$

Now the expression for pressure in Eqn 39 is complete.

Nondimensionalized Pressure and Velocity

As before, unitless expressions are preferred, namely pressure should be normalized by the amplitude of the incident pressure wave A and velocity normalized by $\frac{A}{\rho_0 c}$. First the expressions for U_a and U_b in Eqns 46 and 47 are normalized by $\frac{A}{\rho_0 c}$. These expressions can be substituted into those of normalized pressure and velocity (Eqns 50 and 51). k_{x_m} becomes normalized as well (Eqn 52), a derivation of which is included in the appendix. Within these new expressions are other normalized parameters:

$\bar{Z}_a = \frac{Z_{m_a}}{\frac{H\rho_0 c}{2}}$, $\bar{Z}_b = \frac{Z_{m_b}}{\frac{H\rho_0 c}{2}}$, $\bar{k} = kH$, $\bar{k}_{x_n} = Hk_{x_n}$, $\bar{\omega} = \frac{\omega}{\omega_{N_a}}$, $\bar{\omega}_{c/o} = \frac{\pi c}{H\omega_{N_a}}$, $\bar{t} = t\omega_{N_a}$, $\bar{x} = \frac{x}{H}$, $\bar{y} = \frac{y}{H}$

$$\bar{U}_a = \frac{2\left(\bar{Z}_b + \sum_{n=1}^{\infty}\frac{16\bar{k}}{(2m-1)^2\pi^2\bar{k}_{x_m}}\right)}{\bar{Z}_a\bar{Z}_b + \left(\bar{Z}_a + \bar{Z}_b\right)\left(1 + \sum_{n=1}^{\infty}\frac{8\bar{k}}{(2m-1)^2\pi^2\bar{k}_{x_m}}\right) + \sum_{n=1}^{\infty}\frac{32\bar{k}}{(2m-1)^2\pi^2\bar{k}_{x_m}}} \tag{48}$$

$$\bar{U}_b = \frac{2\left(\bar{Z}_a + \sum_{n=1}^{\infty}\frac{16\bar{k}}{(2m-1)^2\pi^2\bar{k}_{x_m}}\right)}{\bar{Z}_a\bar{Z}_b + \left(\bar{Z}_a + \bar{Z}_b\right)\left(1 + \sum_{n=1}^{\infty}\frac{8\bar{k}}{(2m-1)^2\pi^2\bar{k}_{x_m}}\right) + \sum_{n=1}^{\infty}\frac{32\bar{k}}{(2m-1)^2\pi^2\bar{k}_{x_m}}} \tag{49}$$

$$\bar{p} = \frac{\bar{U}_a + \bar{U}_b}{2}e^{i(\bar{\omega}\bar{t} - \bar{k}\bar{x})} + \sum_{m=1}^{\infty}\frac{2\bar{k}}{(2m-1)\pi\bar{k}_{x_m}}(-1)^{m+1}\left(\bar{U}_b - \bar{U}_a\right)\cos\left((2m-1)\pi\bar{y}\right)e^{i\left(\bar{\omega}\bar{t} - \bar{k}_{x_m}\bar{x}\right)} \tag{50}$$

$$\bar{U} = \frac{\bar{U}_a + \bar{U}_b}{2}e^{i(\bar{\omega}\bar{t} - \bar{k}\bar{x})} + \sum_{n=1}^{\infty}\frac{2}{(2n-1)\pi}(-1)^{n+1}\left(\bar{U}_b - \bar{U}_a\right)\cos\left((2n-1)\pi\bar{y}\right)e^{i\left(\bar{\omega}\bar{t} - \bar{k}_{x_n}\bar{x}\right)} \tag{51}$$

$$\bar{k}_{x_m} = -i\pi\left(\bar{\omega}_{c/o}\right)^{-1}\sqrt{(2m-1)^2\bar{\omega}_{c/o}^2 - \bar{\omega}^2} \tag{52}$$

In the far field ($\bar{x} >>> 0$), the summations in $\bar{p}$ and $\bar{U}$ become finite. (Note that the summations in $\bar{U}_a$ and $\bar{U}_b$ are always infinite.) This is because of the $e^{-i\bar{k}_{x_m}\bar{x}}$ term: $\bar{k}_{x_m}$ can be either real or imaginary depending on the modal value m (for a given $\bar{\omega}$). In the event that $\bar{k}_{x_m}$ is imaginary, $e^{-i\bar{k}_{x_m}\bar{x}}$ becomes a decaying exponential, $e^{-|\bar{k}_{x_m}|\bar{x}}$. At large $\bar{x}$ values, therefore, the exponential term goes to zero and modes that make $\bar{k}_{x_m}$ imaginary do not contribute to the pressure and velocity of the fluid and we say that these modes do not radiate. To find expressions for pressure and velocity that radiate in the far-field, the summations must be truncated according to the condition:

$$m_{\text{rad}} < \frac{\bar{\omega}}{2\bar{\omega}_{c/o}} + \frac{1}{2} \tag{53}$$

As introduced in the appendix's derivation of Eqn 52, $\bar{\omega}_{c/o}$ indicates the frequency at which the first higher mode "turns on" (beause for $\bar{\omega} < \bar{\omega}_{\text{c/o}}$, $\bar{k}_{x_m}$ is imaginary and no higher modes radiate).

Power Radiated Downstream

Power is the product of the real components of $\bar{p}$ and $\bar{U}$ [8]. Those real components are

$$\bar{p}_{\text{Real}} = \left|\frac{\bar{U}_a + \bar{U}_b}{2}\right| \cos\left(\bar{\omega}\bar{t} - \bar{k}\bar{x} + \phi_1\right) + \sum_{m_{\text{rad}}} \frac{2\bar{k}}{(2m-1)\pi\bar{k}_{x_m}} (-1)^{m+1} \left|\bar{U}_b - \bar{U}_a\right| \cos\left((2m-1)\pi\bar{y}\right) \cos\left(\bar{\omega}\bar{t} - \bar{k}_{x_m}\bar{x} + \phi_2\right) \tag{54}$$

$$\bar{U}_{\text{Real}} = \left|\frac{\bar{U}_a + \bar{U}_b}{2}\right| \cos\left(\bar{\omega}\bar{t} - \bar{k}\bar{x} + \phi_1\right) + \sum_{n_{\text{rad}}} \frac{2}{(2n-1)\pi} (-1)^{n+1} \left|\bar{U}_b - \bar{U}_a\right| \cos\left((2n-1)\pi\bar{y}\right) \cos\left(\bar{\omega}\bar{t} - \bar{k}_{x_n}\bar{x} + \phi_2\right) \tag{55}$$

ϕ_1 and ϕ_2 are phase factors of $\frac{\bar{U}_a+\bar{U}_b}{2}$ and $\bar{U}_b - \bar{U}_a$, respectively, but they will disappear after taking the time and spatial average of the power. Power is the product of Eqns 54 and 55. It is important that the summations have different indices so that cross terms are included.

$$\begin{aligned}
\text{Power} = \Pi = &\left(\left|\frac{\bar{U}_a + \bar{U}_b}{2}\right| \cos\left(\bar{\omega}\bar{t} - \bar{k}\bar{x} + \phi_1\right)\right)^2 \\
&+ \left|\frac{\bar{U}_a + \bar{U}_b}{2}\right| \cos\left(\bar{\omega}\bar{t} - \bar{k}\bar{x} + \phi_1\right) \sum_{m_{\text{rad}}} \frac{2\bar{k}}{(2m-1)\pi\bar{k}_{x_m}} (-1)^{m+1} \left|\bar{U}_b - \bar{U}_a\right| \cos\left((2m-1)\pi\bar{y}\right) \cos\left(\bar{\omega}\bar{t} - \bar{k}_{x_m}\bar{x} + \phi_2\right) \\
&+ \left|\frac{\bar{U}_a + \bar{U}_b}{2}\right| \cos\left(\bar{\omega}\bar{t} - \bar{k}\bar{x} + \phi_1\right) \sum_{n_{\text{rad}}} \frac{2}{(2n-1)\pi} (-1)^{n+1} \left|\bar{U}_b - \bar{U}_a\right| \cos\left((2n-1)\pi\bar{y}\right) \cos\left(\bar{\omega}\bar{t} - \bar{k}_{x_n}\bar{x} + \phi_2\right) \\
&+ \sum_{m_{\text{rad}}} \sum_{n_{\text{rad}}} \frac{4\bar{k}}{(2n-1)(2m-1)\pi^2\bar{k}_{x_m}} (-1)^{m+n+2} \left|\bar{U}_b - \bar{U}_a\right|^2 \cos\left((2n-1)\pi\bar{y}\right) \cos\left((2m-1)\pi\bar{y}\right) \\
&\qquad \cos\left(\bar{\omega}\bar{t} - \bar{k}_{x_n}\bar{x} + \phi_2\right) \cos\left(\bar{\omega}\bar{t} - \bar{k}_{x_m}\bar{x} + \phi_2\right)
\end{aligned} \tag{56}$$

First the power is averaged spatially over $\bar{y}$. The height of the duct is from 0 to 1 in the unitless $\bar{y}$. The result of the spatial average is averaged in time over some temporal period τ. The x dependence disappears after these two averages, so a spatial average in the $\hat{x}$ direction is not necessary. The following equations show the sequence of this calculation.

$$\langle \Pi \rangle_{\bar{y}} = \frac{1}{1} \int_0^1 \Pi \; \mathrm{d}\bar{y} \tag{57}$$

$$\left\langle \langle \Pi \rangle_{\bar{y}} \right\rangle_{\bar{t}} = \frac{1}{\tau} \int_{t'}^{t'+\tau} \langle \Pi \rangle_{\bar{y}} \; \mathrm{d}\bar{t} \tag{58}$$

$$\left\langle \langle \Pi \rangle_{\bar{y}} \right\rangle_{\bar{t}} = \frac{1}{4} \left| \bar{U}_a + \bar{U}_b \right|^2 + \sum_{m_{rad}} \frac{2\bar{k} \left| \bar{U}_a - \bar{U}_b \right|^2}{(2m-1)^2 \pi^2 \bar{k}_{x_m}} \tag{59}$$

3 Infinitely Tall Stack of Panels

Now that the behavior of two oscillating panels is understood, the analysis can proceed to examine the desired set-up, namely a barrier composed of infinitely many panels as shown in Figure 1. For simplicity, the depth of the barrier in the $\hat{z}$ direction is neglected and the two-dimensional calculation is considered only in the x-y plane. The view showed to the right in Figure 1 is the angle of interest: The set-up consists of an infinitely tall stack of panels which oscillate in the $\hat{x}$ direction. Note that the origin lies at the center of a panel length.

3.1 Case of Constrained Panel Length

At this stage in the calculation, the direction of propagation of the sound waves is no longer limited to normal incidence and they strike the panels at oblique angles (viewed from the x-y plane). The panels, in response to the incident pressure wave, now oscillate out of phase from one another, even if they have the same dynamic properties such as mass, structural damping, natural frequency, etc. (Identical panels will oscillate with the same displacement amplitude, however.) As the panels oscillate about the y-axis (the red dashed line in Figure 6) out of phase, a wave pattern is created in the $\hat{y}$ direction. This pattern is outlined by the black line in Figure 6.

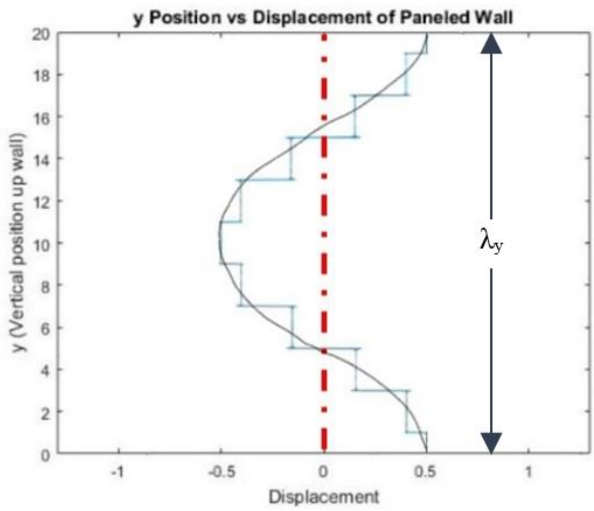

Figure 6: This is a screenshot of an animation in which panels (each vertical, blue segment) oscillate in the horizontal direction, corresponding to the $\hat{x}$ direction as shown in Figure 1. The panels oscillate about the red dashed line (the y-axis). The black line traces the shape of the wave pattern in the $\hat{y}$ direction created by the panels moving out of phase from each other.

The acoustic wavenumber in the $\hat{y}$ direction is k_y. First, the system is constrained such that the wavelength of this pattern, $\lambda_y = \frac{2\pi}{k_y}$, is an integer multiple of the panel height H as shown in Figure 7. In other words, an integer number of panels can "fit" within a vertical wave pattern.

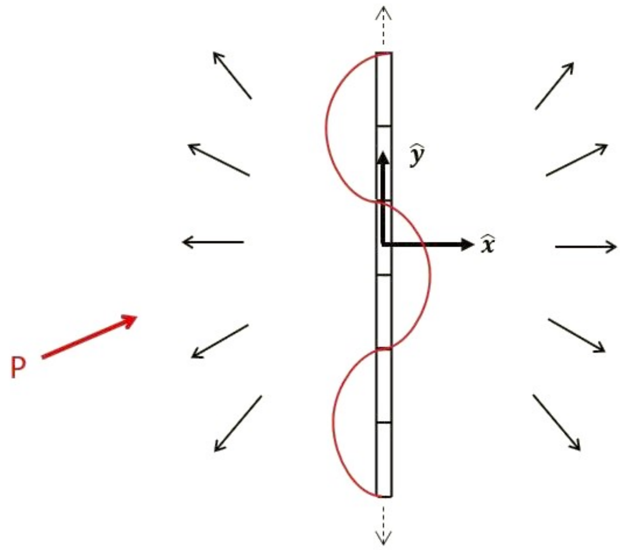

Figure 7: Here the constrained case is shown in which λ_y is an integer multiple of the wavelength (in this diagram, four panels fit in one wavelength). The incident pressure wave P has oblique incidence.

For a given oscillating panel, there are panels oscillating in phase with it every λ_y incremental distance above and below along the wall. Since the displacement of the boundary is periodic, it can be modally decomposed and re-expressed with a complex Fourier series. The motivation for finding the displacement of the barrier is that then the velocity of the barrier and thus the velocity of the fluid at $x = 0$ can be known. This serves as a boundary condition just as when, for the case of panels in a duct, it was established that the velocity of the boundary was piecewise in terms of U_a and U_b.

To find the displacement of the panels in response to the sound source, first the force on a single panel of height H, such as the one centered about the origin, is determined. At the position of the panel or at $x = 0$, the incident pressure wave has the form $p = Pe^{i(\omega t - k\sin\theta)}$ where P is the pressure amplitude. The force on this single panel is

$$f = \int_{\frac{-H}{2}}^{\frac{H}{2}} p\,\mathrm{d}y = \int_{\frac{-\mathrm{H}}{2}}^{\frac{\mathrm{H}}{2}} \mathrm{P}e^{\mathrm{i}(\omega \mathrm{t} - \mathrm{k}\sin\theta)}\,\mathrm{dy} = \frac{2\mathrm{P}}{\mathrm{k}\sin\theta}\sin\left(\frac{\mathrm{kH}\sin\theta}{2}\right)\mathrm{e}^{\mathrm{i}\omega \mathrm{t}} = \mathrm{Fe}^{\mathrm{i}\omega \mathrm{t}} \tag{60}$$

$$F = \frac{2P}{k\sin\theta}\sin\left(\frac{kH\sin\theta}{2}\right)$$

The displacement, x, of this panel due to this force can be determined with a force balance expression. Eqn 61 uses the same parameters for panel dynamics as were used in the previous section (panels in a duct).

$$m\ddot{x} + R\dot{x} + s_c x = Fe^{i\omega t} \tag{61}$$

Assuming a solution of the form $x = Xe^{i\omega t}$, it follows that this panel's displacement is

$$x = \left(-m\omega^2 + iR\omega + s_c\right)^{-1} Fe^{i\omega t} = \alpha F e^{i\omega t} \tag{62}$$

$$\alpha = \left(-m\omega^2 + iR\omega + s_c\right)^{-1}$$

The center of the next adjacent panel is a distance $\frac{\lambda_y}{N} = \frac{2\pi}{Nk\sin\theta}$ away from the center of the previous panel if there are N panels that fit within one λ_y. It follows then that the force on this second panel will be shifted by a phase factor $e^{-i\phi}$ and the n^{th} next panel is phase shifted by $e^{-i\phi_n}$ where $\phi_n = n\phi = \frac{2\pi H n}{\lambda_y} = nkH\sin\theta$. The displacement will have an identical phase shift. For the N number of panels per λ_y, the barrier's displacement can be written as a piecewise function.

$$\eta(y,t) = \begin{cases} \alpha F e^{i\omega t} & n = 0 \\ \alpha F e^{i(\omega t - \phi_1)} & n = 1 \\ \alpha F e^{i(\omega t - \phi_2)} & n = 2 \\ \vdots & \\ \alpha F e^{i(\omega t - \phi_{N-1})} & n = N-1 \end{cases} \tag{63}$$

The formulas for the complex Fourier representation $g(y)$ of a function $f(y)$ with period 2L are

$$g(y) = \sum_{m=-\infty}^{m=\infty} c_m e^{\frac{im\pi y}{L}} \tag{64}$$

$$c_m = \frac{1}{2L}\int_{-L}^{L} f(y)\, e^{\frac{im\pi y}{L}}\, \mathrm{dy} \tag{65}$$

η's spatial period is $\lambda_y = \frac{2\pi}{k\sin\theta}$ so $L = \frac{\pi}{k\sin\theta}$. Since η is a piecewise function, the integral in Eqn 65 becomes a sum of integrals:

$$\begin{aligned} c_m &= \frac{k\sin\theta}{2\pi}\int_{\frac{-\pi}{k\sin\theta}}^{\frac{\pi}{k\sin\theta}} \alpha F e^{i(\omega t - \phi_n - mk\sin\theta y)}\, \mathrm{dy} \\ &= \frac{k\sin\theta}{2\pi}\sum_{n=0}^{N-1}\left[\int_{\frac{\pi(2n-N)}{kN\sin\theta}}^{\frac{\pi(2n-N+2)}{kN\sin\theta}} \alpha F e^{i(\omega t - \phi_n - mk\sin\theta y)}\, \mathrm{dy}\right] \\ &= \frac{\alpha F}{\pi m}\sum_{n=0}^{N-1}\sin\left(\frac{m\pi}{N}\right) e^{i\left(\omega t - \phi_n - \frac{m\pi}{N}(2n-N+1)\right)} \end{aligned} \tag{66}$$

$$\begin{aligned} \eta(y,t) &= \sum_{m=-\infty}^{m=\infty} c_m e^{imk\sin\theta y} \\ &= \sum_{m=-\infty}^{m=\infty} \frac{\alpha F}{\pi m}\sum_{n=0}^{N-1}\sin\left(\frac{m\pi}{N}\right) e^{-i\left(\phi_n + \frac{m\pi}{N}(2n-N+1)\right)} e^{i(\omega t + mk\sin\theta y)} \\ &= \sum_{m=-\infty}^{m=\infty} D_m\, e^{i(\omega t + mk\sin\theta\, y)} \end{aligned} \tag{67}$$

$$D_m = \frac{\alpha F}{\pi m}\sum_{n=0}^{N-1}\sin\left(\frac{m\pi}{N}\right) e^{-i\left(\phi_n + \frac{m\pi}{N}(2n-N+1)\right)}$$

The last line of Eqn 67 shows that the displacement of the barrier is a sum of sinusoidal functions.

The next goal is to find the pressure wave that radiates away from a barrier that has a generic sinusoidal displacement $Ae^{i\left(\omega t-k_y y\right)}$. Then, the pressure radiated away from $\eta\,(y,t)$ will be the sum of those modal pressures.

If the displacement is of the form $De^{i\left(\omega t-k_y y\right)}$, then the velocity of the barrier and thus the fluid at $x=0$ is $U\Big|_{x=0} = \frac{d\eta}{dt} = i\omega De^{i\left(\omega t-k_y y\right)}$. This is the boundary condition that $U\,(x,y,t)$ will meet at $x=0$. As in all cases, the wave and momentum equations are used to find expressions for pressure and velocity. The results are:

$$p\,(x,y,t) = Pe^{i\left(\omega t-k_x x-k_y y\right)} \tag{68}$$

$$U\,(x,y,t) = \frac{P}{\rho_0 c}\frac{k_x}{k}e^{i\left(\omega t-k_x x-k_y y\right)} \tag{69}$$

The boundary condition on velocity requires that

$$\frac{P}{\rho_0 c}\frac{k_x}{k}e^{i\left(\omega t-k_y y\right)} = i\omega De^{i\left(\omega t-k_y y\right)} \tag{70}$$

Solving this equation for the pressure amplitude P yields

$$P = \frac{i\omega\rho_0 ck}{k_x}D \tag{71}$$

Substituting this amplitude into Eqn 68 indicates that when the barrier velocity is of the form $De^{i\left(\omega t-k_y y\right)}$, the pressure at $x>0$ is

$$p = \frac{i\omega\rho_0 ck}{k_x}De^{i\left(\omega t-k_x x-k_y y\right)} \tag{72}$$

It directly follows then that the pressure at $x>0$ produced by the displacement of the infinite stack of panels is

$$\begin{aligned} p\,(x,y,t) &= \sum_{m=-\infty}^{\infty} p_m = \sum_{m=-\infty}^{\infty} \frac{i\omega\rho_0 ck}{k_{x_m}} D_m e^{i\left(\omega t-k_{x_m}x-k_{y_m}y\right)} \\ &= \sum_{m=-\infty}^{\infty} \frac{i\omega\rho_0 ck}{k_x}\frac{\alpha F}{\pi m}\sum_{n=0}^{N-1}\sin\left(\frac{m\pi}{N}\right)e^{-i\left(\phi_n+\frac{m\pi}{N}(2n-N+1)\right)}e^{i\left(\omega t-k_x x-k_y y\right)} \end{aligned} \tag{73}$$

For this calculation, all of the panels are identical; they have the same structural properties such as mass, natural frequency, etc. If we want to investigate the case where every other panel is identical (i.e. the infinitely tall stack of panels alternates between A and B type panels), then the expressions for the barrier's displacement and the pressure in Eqns 67 and 73 can be easily modified. For both of these expressions, the sum with index n iterates through the N panels that fit within a wavelength λ_y. If N is chosen to be an even number, then $\frac{N}{2}$ number of A and B panels each fit within a wavelength. According to Eqn 63, the variable α stores the information about the dynamic properties of panel A, so an analogous term β is introduced which holds the properties of panel B. The expressions become:

$$\begin{aligned} \eta_{AB}\,(y,t) = \sum_{m=-\infty}^{m=\infty}\frac{F}{\pi m}\Bigg[&\sum_{n_{\text{even}}=0}^{N-2}\alpha\sin\left(\frac{m\pi}{N}\right)e^{-i\left(\phi_n+\frac{m\pi}{N}(2n-N+1)\right)}e^{i(\omega t+mk\sin\theta y)} \\ &+\sum_{n_{\text{odd}}=1}^{N-1}\beta\sin\left(\frac{m\pi}{N}\right)e^{-i\left(\phi_n+\frac{m\pi}{N}(2n-N+1)\right)}e^{i(\omega t+mk\sin\theta y)}\Bigg] \end{aligned} \tag{74}$$

$$p_{AB}(x,y,t) = \sum_{m=-\infty}^{\infty} \frac{i\omega\rho_0 ck}{k_x}\frac{F}{\pi m}\left[\sum_{n_{\text{even}}=0}^{N-2} \alpha \sin\left(\frac{m\pi}{N}\right) e^{-i\left(\phi_n + \frac{m\pi}{N}(2n-N+1)\right)} e^{i\left(\omega t - k_x x - k_y y\right)} \right.$$
$$\left. + \sum_{n_{\text{odd}}=1}^{N-1} \beta \sin\left(\frac{m\pi}{N}\right) e^{-i\left(\phi_n + \frac{m\pi}{N}(2n-N+1)\right)} e^{i\left(\omega t - k_x x - k_y y\right)}\right] \tag{75}$$

For the case discussed in this section in which the panel height is constrained (λ_y is an integer multiple of H), the analysis ends at an expression for pressure instead of continuing on to find the power. This is because this special case is useful for verifying that the more generalized, unconstrained, case is correct (if it can reproduce the pressure found here for certain panel height values), but the power radiated downstream for this constrained case is not of primary interest. In the next section, the process of finding the pressure (and eventually power) radiated away from the panels is repeated, but for a generalized case in which the panels can have any height.

3.2 Case of Generalized Panel Length

3.2.1 Identical Panels

Finding x = 0 Boundary Condition, Velocity of Acoustic Barrier:

Now a more generalized case is analyzed where λ_y is no longer necessarily an integer multiple of the panel height H as shown in Figure 8. Just as before in the case of contrained panel height, the first goal is to find an expression for the displacement of the vertical stack of panels.

At this point in the calculation, all of the panels are identical (same impedance and natural frequency) and thus the panels will all oscillate with the same displacement amplitude, just out of phase from each other. Let the displacement have amplitude A (not to be confused with the pressure amplitude A in the section Panels in a Duct). The calculation begins with the assumption that A is known, but later an expression for A will be found and substituted back in. A single panel at its maximum displacement position can be modeled with a heaviside step function. For example, the panel centered about the origin (and thus with start and end y-coordinates of $-\frac{H}{2}$ and $\frac{H}{2}$, respectively) has displacement $A\left[u\left(y+\frac{H}{2}\right) - u\left(y-\frac{H}{2}\right)\right]$ where $u(y)$ is the heaviside step function. Now to account for infinitely many panels where each panel is out of phase from the previous one by a factor of $e^{-ikH\sin\theta}$, the displacement η is:

$$\eta(y,t) = \sum_{n=-\infty}^{\infty} A\left[u\left(y-(n-1/2)H\right) - u\left(y-(n+1/2)H\right)\right] e^{i(\omega t - nkH\sin\theta)} \tag{76}$$

In order to be represented in a more useful form, $\eta(y,t)$ should be modally decomposed and expressed as a sum of periodic fuctions. This is the motivation for following manipulations of the current expression for $\eta(y,t)$. The $e^{i\omega t}$ term is temporarily dropped and the displacement function given in a snapshot of time. A spatial derivative yields:

$$\frac{d\eta(y)}{dy} = \sum_{n=-\infty}^{\infty} A\left[\delta\left(y-(n-1/2)H\right) - \delta\left(y-(n+1/2)H\right)\right] e^{-inkH\sin\theta} \tag{77}$$

Redefining the index n in the second term to $n-1$ gives:

$$\frac{d\eta(y)}{dy} = \sum_{n=-\infty}^{\infty} A\delta\left(y-(n-1/2)H\right)\left[e^{-ikHn\sin\theta} - e^{-ikH(n-1)\sin\theta}\right] \tag{78}$$

Factoring out an exponential term and then using the relation between sine and exponential functions yields:

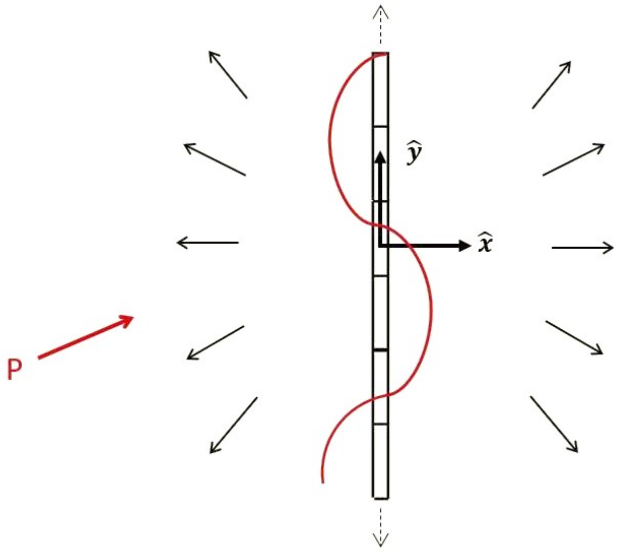

Figure 8: Here the generalized case is shown in which λ_y does not need to be an integer multiple of the wavelength. The incident pressure wave P has oblique incidence.

$$\frac{d\eta(y)}{dy} = \sum_{n=-\infty}^{\infty} -2iA\delta\left(y - (n - 1/2)H\right) e^{-ikH(n-1/2)\sin\theta} \sin\left(\frac{kH\sin\theta}{2}\right) \tag{79}$$

Since the delta functions appear at $y = (n - 1/2)H$ and are zero at all other y, the exponential line in Eqn 79 can changed to include y without affecting the result.

$$\frac{d\eta(y)}{dy} = -2iA\sin\left(\frac{kH\sin\theta}{2}\right) e^{-ik\sin\theta y} \sum_{n=-\infty}^{\infty} \delta\left(y - (n - 1/2)H\right) \tag{80}$$

The summation is a delta comb or an infinite, periodic succession of delta functions. This delta comb has a period of H and can be re-expressed with the complex Fourier series (see Eqns 64 and 65). Including this result, $\frac{d\eta(y)}{dy}$ becomes:

$$\frac{d\eta(y)}{dy} = -\frac{2iA}{H}\sin\left(\frac{kH\sin\theta}{2}\right) \sum_{n=-\infty}^{\infty} (-1)^n e^{i\left(\frac{2\pi n}{H} - k\sin\theta\right)y} \tag{81}$$

Integrating with respect to y and reincorporating the $e^{i\omega t}$ term yields the following function for the collective displacement of all the panels. Note that it is a modal sum of periodic functions, just as was desired.

$$\eta(y,t) = -\frac{2A}{H}\sin\left(\frac{kH\sin\theta}{2}\right) \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{2\pi n}{H} - k\sin\theta\right)^{-1} e^{i\left[\omega t + \left(\frac{2\pi n}{H} - k\sin\theta\right)y\right]} \tag{82}$$

Taking the derivative with respect to time gives the velocity of the panels and thus the velocity of the fluid at $x = 0$. Therefore the function in Eqn 83 is a boundary condition that the expression for velocity (with x dependence) must match. Recall that at this point A is assumed to be known, but must be later defined.

$$\frac{d\eta(y,t)}{dt} = -\frac{2iA\omega}{H}\sin\left(\frac{kH\sin\theta}{2}\right)\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{2\pi n}{H} - k\sin\theta\right)^{-1} e^{i\left[\omega t + \left(\frac{2\pi n}{H} - k\sin\theta\right)y\right]} \tag{83}$$

Finding Acoustic Pressure and Velocity as functions of x, y, and t:

As has been done before, the acoustic pressure and velocity downstream ($x > 0$) are found from the two-dimensional wave and momentum equations. The results are given in Eqns 84 and 85 where P' is a pressure amplitude that must eventually be defined (The "prime" notation distinguishes this P from another that will be used shortly).

$$p(x,y,t) = P' e^{i(\omega t - k_x x - k_y y)} \tag{84}$$

$$u(x,y,t) = \frac{P'}{\rho_0 c}\frac{k_x}{k} e^{i(\omega t - k_x x - k_y y)} \tag{85}$$

Now using the boundary condition for velocity at $x = 0$, Eqn 85 is related to Eqn 83 as follows:

$$\frac{P'}{\rho_0 c}\frac{k_x}{k} e^{i(\omega t - k_y y)} = -\frac{2iA\omega}{H}\sin\left(\frac{kH\sin\theta}{2}\right)\sum_{n=-\infty}^{\infty}(-1)^n\left(\frac{2\pi n}{H} - k\sin\theta\right)^{-1} e^{i\left[\omega t + \left(\frac{2\pi n}{H} - k\sin\theta\right)y\right]} \tag{86}$$

From the exponential terms it follows that $k_{y_n} = k\sin\theta - \frac{2\pi n}{H}$. Solving for the pressure amplitude P' gives

$$P' = -\frac{2iA\omega\rho_0 ck}{H}\sin\left(\frac{kH\sin\theta}{2}\right)\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{k_{x_n}}\left(\frac{2\pi n}{H} - k\sin\theta\right)^{-1} \tag{87}$$

Now substituting this expression into Eqn 84 yields the following expression for pressure

$$p(x,y,t) = -\frac{2iA\omega\rho_0 ck}{H}\sin\left(\frac{kH\sin\theta}{2}\right)\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{k_{x_n}}\left(\frac{2\pi n}{H} - k\sin\theta\right)^{-1} e^{i\left[\omega t - k_{x_n}x - \left(k\sin\theta - \frac{2\pi n}{H}\right)y\right]} \tag{88}$$

Finding Expression for Displacement Amplitude A:

Now there is sufficient information to find an expression for the displacement amplitude A included in the Eqn 88. The outline for finding A is to first find the total force (a function of A) on a single panel. Then this force is related to the panel's velocity (also a function of A) through the mechanical impedance. A can be isolated from this relation so that it is found as a function of both impedance and the amplitude of the sound source's pressure amplitude.

As before, the total force on a panel can be found using the principle of superposition: the total force is the sum of the "blocked force" and the force due to panel motion. Now, however, the blocked force is due to a pressure wave at oblique incidence. Let this incoming pressure wave, at $x = 0$, be $p = Pe^{i(\omega t - k_y y)}$. Then the blocked force is given in Eqn 89 where the chosen panel is centered at the origin (extends from $-\frac{H}{2}$ to $\frac{H}{2}$) and the factor of 2 in front of the integral accounts for contribution from the reflected wave.

$$F_{\text{Blocked}} = 2\int_{-\frac{H}{2}}^{\frac{H}{2}} p\,dy = 2\int_{-\frac{H}{2}}^{\frac{H}{2}} Pe^{i(\omega t - k_y y)}\,dy = \frac{2iP}{k_y}e^{i(\omega t - k_y y)}\Big|_{-\frac{H}{2}}^{\frac{H}{2}} = \frac{4P}{k\sin\theta}\sin\left(\frac{kH\sin\theta}{2}\right)e^{i\omega t} \tag{89}$$

To find the backloading contribution to the total force, the pressure at $x = 0$ is integrated over the panel length. The expression for pressure is the one in Eqn 88. Again, the -2 factor before the integral accounts for the phenomena in which the pressure "pushes" from the right and "pulls" from the left, doubling the force in the $-\hat{x}$ direction.

$$\begin{aligned} F_{\text{Backloading}} = -2\int_{-\frac{H}{2}}^{\frac{H}{2}} p\,dy = \int_{-\frac{H}{2}}^{\frac{H}{2}} \frac{4iA\omega\rho_0 ck}{H}\sin\left(\frac{kH\sin\theta}{2}\right)\sum_{m=-\infty}^{m=\infty}\frac{(-1)^m}{k_{x_m}}\left(\frac{2\pi m}{H} - k\sin\theta\right)^{-1} \\ e^{i\left(\omega t - \left(k\sin\theta - \frac{2\pi m}{H}\right)y\right)} \qquad (90) \\ = \frac{8iA\omega\rho_0 ck}{H}\sin\left(\frac{kH\sin\theta}{2}\right)\sum_{m=-\infty}^{m=\infty}\frac{(-1)^m}{k_{x_m}}\left(\frac{2\pi m}{H} - k\sin\theta\right)^{-2}\sin\left(\pi m - \frac{kH\sin\theta}{2}\right)e^{i\omega t} \end{aligned}$$

Now using superposition, i.e. $F_{\text{Total}} = F_{\text{Blocked}} + F_{\text{Backloading}}$, the total force on a single panel is

$$F_{\text{Total}} = \sin\left(\frac{kH\sin\theta}{2}\right)\left[\frac{4P}{k\sin\theta} + \frac{8iA\omega\rho_0 ck}{H}\sum_{m=-\infty}^{m=\infty}\frac{(-1)^m}{k_{x_m}}\left(\frac{2\pi m}{H} - k\sin\theta\right)^{-2}\sin\left(\pi m - \frac{kH\sin\theta}{2}\right)\right]e^{i\omega t} \tag{91}$$

The velocity u of this single panel with displacement amplitude A is $Ai\omega e^{i\omega t}$ (just the time derivative of the panel displacement $Ae^{i\omega t}$). The mechanical impedance, defined as the ratio between net force and velocity of the panel, thus is:

$$Z_m = \frac{F_{\text{Total}}}{u} = \sin\left(\frac{kH\sin\theta}{2}\right)\left[\frac{4P}{iA\omega k\sin\theta} + \frac{8\rho_0 ck}{H}\sum_{m=-\infty}^{m=\infty}\frac{(-1)^m}{k_{x_m}}\left(\frac{2\pi m}{H} - k\sin\theta\right)^{-2}\sin\left(\pi m - \frac{kH\sin\theta}{2}\right)\right] \tag{92}$$

Now if this expression is rearranged and solved for A, the displacement amplitude is

$$A = \frac{4P}{ik\omega\sin\theta}\left[\frac{Z_m}{\sin\left(\frac{kH\sin\theta}{2}\right)} - \frac{8\rho_0 ck}{H}\sum_{m=-\infty}^{m=\infty}\frac{(-1)^m}{k_{x_m}}\left(\frac{2\pi m}{H} - k\sin\theta\right)^{-2}\sin\left(\pi m - \frac{kH\sin\theta}{2}\right)\right]^{-1} \tag{93}$$

Nondimensionalized Pressure and Velocity:

Now that A is known, Eqn 93 is substituted into 88 and pressure re-expressed as:

$$\begin{aligned} p(x,y,t) = \frac{-8P\rho_0 c}{H\sin\theta}\left[\frac{Z_m}{\sin\left(\frac{kH\sin\theta}{2}\right)} - \frac{8\rho_0 ck}{H}\sum_{m=-\infty}^{m=\infty}\frac{(-1)^m}{k_{x_m}}\left(\frac{2\pi m}{H} - k\sin\theta\right)^{-2}\sin\left(\pi m - \frac{kH\sin\theta}{2}\right)\right]^{-1} \\ \sin\left(\frac{kH\sin\theta}{2}\right)\sum_{n=-\infty}^{n=\infty}\frac{(-1)^n}{k_{x_n}}\left(\frac{2\pi n}{H} - k\sin\theta\right)^{-1} e^{i\left[\omega t - k_{x_n}x - \left(k\sin\theta - \frac{2\pi n}{H}\right)y\right]} \end{aligned} \tag{94}$$

This expression for p is then normalized by the amplitude of the incident pressure wave, P. Then it is rewritten in terms of dimensionless parameters. The nondimensionalized expression for pressure is:

$$\bar{p}\left(\bar{x},\bar{y},\bar{t}\right)=\frac{p\left(x,y,t\right)}{P}=\frac{-8}{\sin\theta}\left[\frac{\bar{Z}}{\sin\left(\frac{k_N\bar{\omega}\sin\theta}{2}\right)}-8k_N\bar{\omega}\sum_{m=-\infty}^{m=\infty}\frac{(-1)^m}{\bar{k}_{x_m}}\left(2\pi m-k_N\bar{\omega}\sin\theta\right)^{-2}\sin\left(\pi m-\frac{k_N\bar{\omega}\sin\theta}{2}\right)\right]^{-1}$$
$$\sin\left(\frac{k_N\bar{\omega}\sin\theta}{2}\right)\sum_{n=-\infty}^{n=\infty}\frac{(-1)^n}{\bar{k}_{x_n}}\left(2\pi n-k_N\bar{\omega}\sin\theta\right)^{-1}e^{i\left[\bar{\omega}\bar{t}-\bar{k}_{x_n}\bar{x}-(k_N\bar{\omega}\sin\theta-2\pi n)\bar{y}\right]} \tag{95}$$

$$\bar{Z}=\frac{Z_m}{H\rho_0 c},\ kH=k_N\bar{\omega},\ k_N=\frac{\omega_N H}{c},\ \bar{\omega}=\frac{\omega}{\omega_N},\ \bar{k}_{x_n}=Hk_{x_n},\ \bar{k}_{y_n}=Hk_{y_n},\ \bar{t}=t\omega_N,\ \bar{x}=\frac{x}{H},\ \bar{y}=\frac{y}{H}$$
$$\bar{k}_{x_n}=\sqrt{\bar{k}^2-\bar{k}_{y_n}^2}=\sqrt{(k_N\bar{\omega})^2-(k_N\bar{\omega}\sin\theta-2\pi n)^2}=-i\sqrt{(k_N\bar{\omega}\sin\theta-2\pi n)^2-(k_N\bar{\omega})^2} \tag{96}$$

In Eqn 95, the sum with index m always iterates from $-\infty$ to ∞; it comes from the displacement amplitude A which stores information about all of the modes contributing to the force on the barrier. The sum with index n, however, becomes truncated when the pressure is calculated at a large enough $\bar{x}$ coordinate away from the barrier that only radiating modes remain. As previously mentioned, for a mode to radiate, $\bar{k}_{x_n}$ must be real. It follows that

$$\frac{-k_N\bar{\omega}\left(1-\sin\theta\right)}{2\pi}<m_{\text{Radiating}}<\frac{k_N\bar{\omega}\left(1+\sin\theta\right)}{2\pi} \tag{97}$$

With this condition on the sum with index n in Eqn 95, the radiating nondimensionalized pressure (pressure found in the far field) is:

$$\bar{p}_{\text{Rad}}\left(\bar{x},\bar{y},\bar{t}\right)=\frac{p\left(x,y,t\right)}{P}=\frac{-8}{\sin\theta}\left[\frac{\bar{Z}}{\sin\left(\frac{k_N\bar{\omega}\sin\theta}{2}\right)}-8k_N\bar{\omega}\sum_{m=-\infty}^{m=\infty}\frac{(-1)^m}{\bar{k}_{x_m}}\left(2\pi m-k_N\bar{\omega}\sin\theta\right)^{-2}\sin\left(\pi m-\frac{k_N\bar{\omega}\sin\theta}{2}\right)\right]^{-1}$$
$$\sin\left(\frac{k_N\bar{\omega}\sin\theta}{2}\right)\sum_{n_{\text{Rad}}}\frac{(-1)^n}{\bar{k}_{x_n}}\left(2\pi n-k_N\bar{\omega}\sin\theta\right)^{-1}e^{i\left[\bar{\omega}\bar{t}-\bar{k}_{x_n}\bar{x}-(k_N\bar{\omega}\sin\theta-2\pi n)\bar{y}\right]} \tag{98}$$

As has been calculated throughout this paper, the velocity of the sound waves far away from the panels (such that only radiating modes remain) can be found by relating velocity to pressure with the momentum equation. The nondimensionalized result is given in Eqn 99.

$$\bar{u}_{\text{Rad}}\left(\bar{x},\bar{y},\bar{t}\right)=\frac{u\left(x,y,t\right)}{\frac{P}{\rho_0 c}}=\frac{-8}{k_N\bar{\omega}\sin\theta}\left[\frac{\bar{Z}}{\sin\left(\frac{k_N\bar{\omega}\sin\theta}{2}\right)}-8k_N\bar{\omega}\sum_{m=-\infty}^{m=\infty}\frac{(-1)^m}{\bar{k}_{x_m}}\left(2\pi m-k_N\bar{\omega}\sin\theta\right)^{-2}\sin\left(\pi m-\frac{k_N\bar{\omega}\sin\theta}{2}\right)\right]^{-1}$$
$$\sin\left(\frac{k_N\bar{\omega}\sin\theta}{2}\right)\sum_{n_{\text{Rad}}}(-1)^n\left(2\pi n-k_N\bar{\omega}\sin\theta\right)^{-1}e^{i\left[\bar{\omega}\bar{t}-\bar{k}_{x_n}\bar{x}-(k_N\bar{\omega}\sin\theta-2\pi n)\bar{y}\right]} \tag{99}$$

Power Radiated Downstream:

The real components of the radiating pressure and velocity, given in Eqns 98 and 99, are

$$\bar{p}_{\text{Real}}(\bar{x},\bar{y},\bar{t}) = \frac{-8}{\sin\theta}\left|\left[\frac{\bar{Z}}{\sin\left(\frac{k_N\bar{\omega}\sin\theta}{2}\right)} - 8k_N\bar{\omega}\sum_{m=-\infty}^{m=\infty}\frac{(-1)^m}{\bar{k}_{x_m}}(2\pi m - k_N\bar{\omega}\sin\theta)^{-2}\sin\left(\pi m - \frac{k_N\bar{\omega}\sin\theta}{2}\right)\right]^{-1}\right|$$
$$\sin\left(\frac{k_N\bar{\omega}\sin\theta}{2}\right)\sum_{n_{\text{Rad}}}\frac{(-1)^n}{\bar{k}_{x_n}}(2\pi n - k_N\bar{\omega}\sin\theta)^{-1}\cos\left(\bar{\omega}\bar{t} - \bar{k}_{x_n}\bar{x} - (k_N\bar{\omega}\sin\theta - 2\pi n)\,\bar{y} + \phi\right) \tag{100}$$

$$\bar{u}_{\text{Real}}(\bar{x},\bar{y},\bar{t}) = \frac{-8}{k_N\bar{\omega}\sin\theta}\left|\left[\frac{\bar{Z}}{\sin\left(\frac{k_N\bar{\omega}\sin\theta}{2}\right)} - 8k_N\bar{\omega}\sum_{m=-\infty}^{m=\infty}\frac{(-1)^m}{\bar{k}_{x_m}}(2\pi m - k_N\bar{\omega}\sin\theta)^{-2}\sin\left(\pi m - \frac{k_N\bar{\omega}\sin\theta}{2}\right)\right]^{-1}\right|$$
$$\sin\left(\frac{k_N\bar{\omega}\sin\theta}{2}\right)\sum_{q_{\text{Rad}}}(-1)^q(2\pi q - k_N\bar{\omega}\sin\theta)^{-1}\cos\left(\bar{\omega}\bar{t} - \bar{k}_{x_q}\bar{x} - (k_N\bar{\omega}\sin\theta - 2\pi q)\,\bar{y} + \phi\right) \tag{101}$$

The power radiated downstream is the product of the real components of the radiating pressure and velocity:

$$\Pi_{\text{Rad}} = \frac{64}{k_N\bar{\omega}\sin^2\theta}\left|\left[\frac{\bar{Z}}{\sin\left(\frac{k_N\bar{\omega}\sin\theta}{2}\right)} - 8k_N\bar{\omega}\sum_{m=-\infty}^{m=\infty}\frac{(-1)^m}{\bar{k}_{x_m}}(2\pi m - k_N\bar{\omega}\sin\theta)^{-2}\sin\left(\pi m - \frac{k_N\bar{\omega}\sin\theta}{2}\right)\right]^{-1}\right|^2$$
$$\sin^2\left(\frac{k_N\bar{\omega}\sin\theta}{2}\right)\sum_{n_{\text{Rad}}}\frac{(-1)^n}{\bar{k}_{x_n}}(2\pi n - k_N\bar{\omega}\sin\theta)^{-1}\cos\left(\bar{\omega}\bar{t} - \bar{k}_{x_n}\bar{x} - (k_N\bar{\omega}\sin\theta - 2\pi n)\,\bar{y} + \phi\right)$$
$$\sum_{q_{\text{Rad}}}(-1)^q(2\pi q - k_N\bar{\omega}\sin\theta)^{-1}\cos\left(\bar{\omega}\bar{t} - \bar{k}_{x_q}\bar{x} - (k_N\bar{\omega}\sin\theta - 2\pi q)\,\bar{y} + \phi\right) \tag{102}$$

This time it is mathematically more convenient to perform the average in time $\bar{t}$ first:

$$\langle\Pi\rangle_{\bar{t}} = \frac{1}{\tau}\int_{t'}^{t'+\tau}\Pi\ \mathrm{d}\bar{t} \tag{103}$$

$$\langle\Pi\rangle_{\bar{t}} = \frac{64}{k_N\bar{\omega}\sin^2\theta}\left|\left[\frac{\bar{Z}}{\sin\left(\frac{k_N\bar{\omega}\sin\theta}{2}\right)} - 8k_N\bar{\omega}\sum_{m=-\infty}^{m=\infty}\frac{(-1)^m}{\bar{k}_{x_m}}(2\pi m - k_N\bar{\omega}\sin\theta)^{-2}\sin\left(\pi m - \frac{k_N\bar{\omega}\sin\theta}{2}\right)\right]^{-1}\right|^2$$
$$\sin^2\left(\frac{k_N\bar{\omega}\sin\theta}{2}\right)\sum_{n_{\text{Rad}}}\sum_{q_{\text{Rad}}}\frac{(-1)^{n+q}}{\bar{k}_{x_n}}(2\pi n - k_N\bar{\omega}\sin\theta)^{-1}(2\pi q - k_N\bar{\omega}\sin\theta)^{-1}\cos\left(\left(\bar{k}_{x_q} - \bar{k}_{x_n}\right)\bar{x} + 2\pi(n-q)\,\bar{y}\right) \tag{104}$$

$$\langle\langle\Pi\rangle_{\bar{t}}\rangle_{\bar{y}} = \frac{1}{1}\int_0^1\langle\Pi\rangle_{\bar{t}}\ \mathrm{d}\bar{y} \tag{105}$$

After the spatial average in $\bar{y}$, the power is zero unless $n_{\text{Rad}} = q_{\text{Rad}}$. The final expression for the average radiated pressure downstream therefore has only one sum and is:

$$\langle\langle\Pi\rangle_{\bar{t}}\rangle_{\bar{y}} = \frac{64}{k_N\bar{\omega}\sin^2\theta}\left|\left[\frac{\bar{Z}}{\sin\left(\frac{k_N\bar{\omega}\sin\theta}{2}\right)} - 8k_N\bar{\omega}\sum_{m=-\infty}^{m=\infty}\frac{(-1)^m}{\bar{k}_{x_m}}(2\pi m - k_N\bar{\omega}\sin\theta)^{-2}\sin\left(\pi m - \frac{k_N\bar{\omega}\sin\theta}{2}\right)\right]^{-1}\right|^2$$

$$\sin^2\left(\frac{k_N\bar{\omega}\sin\theta}{2}\right)\sum_{n_{\text{Rad}}}\frac{1}{\bar{k}_{x_n}}(2\pi n - k_N\bar{\omega}\sin\theta)^{-2} \tag{106}$$

$$\bar{k}_{x_n} = -i\sqrt{(k_N\bar{\omega}\sin\theta - 2\pi n)^2 - (k_N\bar{\omega})^2}$$

$$\tfrac{-k_N\bar{\omega}(1-\sin\theta)}{2\pi} < n_{\text{Radiating}} < \tfrac{k_N\bar{\omega}(1+\sin\theta)}{2\pi}$$

$$\bar{Z} = \tfrac{Z_m}{\rho_0 cH} = \tfrac{1}{2\zeta_{aca}}\left[2\zeta_a + i\bar{\omega}\left(1 - \tfrac{1}{\bar{\omega}^2}\right)\right]$$

3.2.2 Alternating Panels

Now the analysis investigates the case in which there are two panel types, A and B, that have different dynamics (mass, natural frequency, etc.). The infinitely tall stack of panels is composed of alternating panels such that every other panel is identical. The previous argument will be repeated with this addition.

The barrier displacement η_{AB} is

$$\begin{aligned}\eta_{AB}(y,t) = &\sum_{n_{\text{odd}}=-\infty}^{\infty} A\left[u(y-(n-1/2)H) - u(y-(n+1/2)H)\right]e^{i(\omega t - nkH\sin\theta)} \\ &+ \sum_{n_{\text{even}}=-\infty}^{\infty} B\left[u(y-(n-1/2)H) - u(y-(n+1/2)H)\right]e^{i(\omega t - nkH\sin\theta)}\end{aligned} \tag{107}$$

A spatial derivative yields:

$$\begin{aligned}\frac{d\eta_{AB}(y,t)}{dy} = &\sum_{n_{\text{odd}}=-\infty}^{\infty} A\left[\delta(y-(n-1/2)H) - \delta(y-(n+1/2)H)\right]e^{i(\omega t - nkH\sin\theta)} \\ &+ \sum_{n_{\text{even}}=-\infty}^{\infty} B\left[\delta(y-(n-1/2)H) - \delta(y-(n+1/2)H)\right]e^{i(\omega t - nkH\sin\theta)}\end{aligned} \tag{108}$$

Redefining the index n in the second term of both sums to $n-1$ gives:

$$\begin{aligned}\frac{d\eta_{AB}(y,t)}{dy} = &\sum_{n_{\text{odd}}=-\infty}^{\infty} \delta(y-(n-1/2)H)\left[Ae^{i(\omega t - nkH\sin\theta)} - Be^{i(\omega t - nkH\sin\theta)}\right] \\ &+ \sum_{n_{\text{even}}=-\infty}^{\infty} \delta(y-(n-1/2)H)\left[Be^{i(\omega t - nkH\sin\theta)} - Ae^{i(\omega t - nkH\sin\theta)}\right]\end{aligned} \tag{109}$$

Factoring out $e^{-i\left(n-\frac{1}{2}\right)kH\sin\theta}$ and replacing it with $e^{-ik\sin\theta y}$ gives:

$$\begin{aligned}\frac{d\eta_{AB}(y,t)}{dy} = e^{i(\omega t - k\sin\theta y)}\Bigg[&\left(Ae^{-ik\sin\theta\frac{H}{2}} - Be^{ik\sin\theta\frac{H}{2}}\right)\sum_{n_{\text{odd}}=-\infty}^{\infty}\delta(y-(n-1/2)H) \\ &+ \left(Be^{-ik\sin\theta\frac{H}{2}} - Ae^{ik\sin\theta\frac{H}{2}}\right)\sum_{n_{\text{even}}=-\infty}^{\infty}\delta(y-(n-1/2)H)\Bigg]\end{aligned} \tag{110}$$

Each summation is a delta comb of period $2H$ and can be re-expressed with the complex Fourier series (see Eqns 64 and 65).

$$\sum_{n_{\text{odd}}=-\infty}^{\infty} \delta\left(y - (n - 1/2)H\right) = \sum_{n=-\infty}^{\infty} \frac{(-i)^n}{2H} e^{\frac{in\pi y}{H}} \tag{111}$$

$$\sum_{n_{\text{even}}=-\infty}^{\infty} \delta\left(y - (n - 1/2)H\right) = \sum_{n=-\infty}^{\infty} \frac{(i)^n}{2H} e^{\frac{in\pi y}{H}} \tag{112}$$

Including these results, $\frac{d\eta_{AB}(y)}{dy}$ becomes:

$$\frac{d\eta_{AB}(y,t)}{dy} = \frac{1}{2H} \sum_{n=-\infty}^{\infty} e^{i\left(\omega t + \left(\frac{n\pi}{H} - k\sin\theta\right)y\right)} \left[\left(Ae^{-ik\sin\theta\frac{H}{2}} - Be^{ik\sin\theta\frac{H}{2}}\right)(-i)^n + \left(Be^{-ik\sin\theta\frac{H}{2}} - Ae^{ik\sin\theta\frac{H}{2}}\right)(i)^n\right] \tag{113}$$

Integrating with respect to y yields the following function for the collective displacement of all the panels.

$$\begin{aligned} \eta_{Ab}(y,t) = \frac{1}{2H} \sum_{n=-\infty}^{\infty} \left(i\left(\frac{n\pi}{H} - k\sin\theta\right)\right)^{-1} e^{i\left(\omega t + \left(\frac{n\pi}{H} - k\sin\theta\right)y\right)} &\left[\left(Ae^{-ik\sin\theta\frac{H}{2}} - Be^{ik\sin\theta\frac{H}{2}}\right)(-i)^n \right.\\ &\left. + \left(Be^{-ik\sin\theta\frac{H}{2}} - Ae^{ik\sin\theta\frac{H}{2}}\right)(i)^n\right] \end{aligned} \tag{114}$$

Taking the derivative with respect to time gives the velocity of the panels and thus the fluid at $x = 0$.

$$\begin{aligned} \frac{d\eta_{AB}(y,t)}{dt} = \frac{\omega}{2H} \sum_{n=-\infty}^{\infty} \left(\frac{n\pi}{H} - k\sin\theta\right)^{-1} e^{i\left(\omega t + \left(\frac{n\pi}{H} - k\sin\theta\right)y\right)} &\left[\left(Ae^{-ik\sin\theta\frac{H}{2}} - Be^{ik\sin\theta\frac{H}{2}}\right)(-i)^n \right.\\ &\left. + \left(Be^{-ik\sin\theta\frac{H}{2}} - Ae^{ik\sin\theta\frac{H}{2}}\right)(i)^n\right] \end{aligned} \tag{115}$$

Finding Acoustic Pressure and Velocity as functions of x, y, and t:

As before,

$$p(x,y,t) = P' e^{i(\omega t - k_x x - k_y y)} \tag{116}$$

$$u(x,y,t) = \frac{P'}{\rho_0 c}\frac{k_x}{k} e^{i(\omega t - k_x x - k_y y)} \tag{117}$$

Now using the boundary condition for velocity at $x = 0$, Eqn 117 is related to Eqn 115 as follows:

$$\begin{aligned} \frac{P'}{\rho_0 c}\frac{k_x}{k} e^{i(\omega t - k_y y)} = \frac{\omega}{2H} \sum_{n=-\infty}^{\infty} \left(\frac{n\pi}{H} - k\sin\theta\right)^{-1} e^{i\left(\omega t + \left(\frac{n\pi}{H} - k\sin\theta\right)y\right)} &\left[\left(Ae^{-ik\sin\theta\frac{H}{2}} - Be^{ik\sin\theta\frac{H}{2}}\right)(-i)^n \right.\\ &\left. + \left(Be^{-ik\sin\theta\frac{H}{2}} - Ae^{ik\sin\theta\frac{H}{2}}\right)(i)^n\right] \end{aligned} \tag{118}$$

From the exponential terms it follows that $k_{y_n} = k\sin\theta - \frac{n\pi}{H}$. Solving for the pressure amplitude P' and substituting it into Eqn 84 yields the following expression for pressure:

$$p_{AB}(x,y,t) = \frac{\omega k \rho_0 c}{2H} \sum_{n=-\infty}^{\infty} \frac{1}{k_{x_n}} \left(\frac{n\pi}{H} - k\sin\theta\right)^{-1} \Big[\left(Ae^{-ik\sin\theta\frac{H}{2}} - Be^{ik\sin\theta\frac{H}{2}}\right)(-i)^n + \left(Be^{-ik\sin\theta\frac{H}{2}} - Ae^{ik\sin\theta\frac{H}{2}}\right)(i)^n\Big] e^{i\left(\omega t - k_{x_n}x - \left(k\sin\theta - \frac{n\pi}{H}\right)y\right)} \tag{119}$$

Velocity is calculated from the momentum equation:

$$U_{AB}(x,y,t) = \frac{\omega}{2H} \sum_{n=-\infty}^{\infty} \left(\frac{n\pi}{H} - k\sin\theta\right)^{-1} \Big[\left(Ae^{-ik\sin\theta\frac{H}{2}} - Be^{ik\sin\theta\frac{H}{2}}\right)(-i)^n + \left(Be^{-ik\sin\theta\frac{H}{2}} - Ae^{ik\sin\theta\frac{H}{2}}\right)(i)^n\Big] e^{i\left(\omega t - k_{x_n}x - \left(k\sin\theta - \frac{n\pi}{H}\right)y\right)} \tag{120}$$

Finding Expression for Displacement Amplitude A:

Now there is enough information to find expressions for the displacement amplitudes A and B included in the Eqn 120. As before, the total force on a panel can be found using the principle of superposition.

$$F_{A_{\text{Blocked}}} = 2\int_{-\frac{H}{2}}^{\frac{H}{2}} p\, dy = \frac{4P}{k\sin\theta} \sin\left(\frac{kH\sin\theta}{2}\right) e^{i\omega t} \tag{121}$$

$$F_{B_{\text{Blocked}}} = 2\int_{\frac{H}{2}}^{\frac{3H}{2}} p\, dy = \frac{4P}{k\sin\theta} \sin\left(\frac{kH\sin\theta}{2}\right) e^{i(\omega t - kH\sin\theta)} \tag{122}$$

$$\begin{aligned} F_{A_{\text{Backloading}}} &= -2\int_{-\frac{H}{2}}^{\frac{H}{2}} p_{AB}\, dy = \int_{-\frac{H}{2}}^{\frac{H}{2}} \frac{-\omega k \rho_0 c}{H} \sum_{m=-\infty}^{\infty} \frac{1}{k_{x_m}} \left(\frac{m\pi}{H} - k\sin\theta\right)^{-1} \Big[\left(Ae^{-ik\sin\theta\frac{H}{2}} - Be^{ik\sin\theta\frac{H}{2}}\right)(-i)^m \\ &\qquad + \left(Be^{-ik\sin\theta\frac{H}{2}} - Ae^{ik\sin\theta\frac{H}{2}}\right)(i)^m\Big] e^{i\left(\omega t - \left(k\sin\theta - \frac{m\pi}{H}\right)y\right)}\, \mathrm{dy} \\ &= \frac{-2\omega k \rho_0 c}{H} \sum_{m=-\infty}^{\infty} \frac{1}{k_{x_m}} \left(\frac{m\pi}{H} - k\sin\theta\right)^{-2} \sin\left(\frac{m\pi}{2} - \frac{kH\sin\theta}{2}\right) \Big[\left(Ae^{-ik\sin\theta\frac{H}{2}} - Be^{ik\sin\theta\frac{H}{2}}\right)(-i)^m \\ &\qquad + \left(Be^{-ik\sin\theta\frac{H}{2}} - Ae^{ik\sin\theta\frac{H}{2}}\right)(i)^m\Big] e^{i\omega t} \end{aligned} \tag{123}$$

$$\begin{aligned} F_{B_{\text{Backloading}}} &= -2\int_{\frac{H}{2}}^{\frac{3H}{2}} p_{AB}\, dy = \int_{\frac{H}{2}}^{\frac{3H}{2}} \frac{-\omega k \rho_0 c}{H} \sum_{m=-\infty}^{\infty} \frac{1}{k_{x_m}} \left(\frac{m\pi}{H} - k\sin\theta\right)^{-1} \Big[\left(Ae^{-ik\sin\theta\frac{H}{2}} - Be^{ik\sin\theta\frac{H}{2}}\right)(-i)^m \\ &\qquad + \left(Be^{-ik\sin\theta\frac{H}{2}} - Ae^{ik\sin\theta\frac{H}{2}}\right)(i)^m\Big] e^{i\left(\omega t - \left(k\sin\theta - \frac{m\pi}{H}\right)y\right)}\, \mathrm{dy} \\ &= \frac{-2\omega k \rho_0 c}{H} \sum_{n=-\infty}^{\infty} \frac{1}{k_{x_m}} \left(\frac{m\pi}{H} - k\sin\theta\right)^{-2} \sin\left(\frac{m\pi}{2} - \frac{kH\sin\theta}{2}\right) \Big[\left(Ae^{-ik\sin\theta\frac{H}{2}} - Be^{ik\sin\theta\frac{H}{2}}\right)(-i)^m \\ &\qquad + \left(Be^{-ik\sin\theta\frac{H}{2}} - Ae^{ik\sin\theta\frac{H}{2}}\right)(i)^m\Big] e^{i(\omega t - kH\sin\theta + m\pi)} \end{aligned} \tag{124}$$

Now using superposition, i.e. $F_{\text{Total}} = F_{\text{Blocked}} + F_{\text{Backloading}}$:

$$\begin{aligned} F_{A_{\text{Total}}} &= \frac{4P}{k\sin\theta} \sin\left(\frac{kH\sin\theta}{2}\right) e^{i\omega t} - \frac{2\omega k \rho_0 c}{H} \sum_{m=-\infty}^{\infty} \frac{1}{k_{x_m}} \left(\frac{m\pi}{H} - k\sin\theta\right)^{-2} \sin\left(\frac{m\pi}{2} - \frac{kH\sin\theta}{2}\right) \\ &\qquad \Big[\left(Ae^{-ik\sin\theta\frac{H}{2}} - Be^{ik\sin\theta\frac{H}{2}}\right)(-i)^m + \left(Be^{-ik\sin\theta\frac{H}{2}} - Ae^{ik\sin\theta\frac{H}{2}}\right)(i)^m\Big] e^{i\omega t} \end{aligned} \tag{125}$$

$$F_{B_{\text{Total}}} = \frac{4P}{k\sin\theta}\sin\left(\frac{kH\sin\theta}{2}\right)e^{i(\omega t - kH\sin\theta)} - \frac{2\omega k\rho_0 c}{H}\sum_{m=-\infty}^{\infty}\frac{1}{k_{x_m}}\left(\frac{m\pi}{H} - k\sin\theta\right)^{-2}\sin\left(\frac{m\pi}{2} - \frac{kH\sin\theta}{2}\right)$$
$$\left[\left(Ae^{-ik\sin\theta\frac{H}{2}} - Be^{ik\sin\theta\frac{H}{2}}\right)(-i)^m + \left(Be^{-ik\sin\theta\frac{H}{2}} - Ae^{ik\sin\theta\frac{H}{2}}\right)(i)^m\right]e^{i(\omega t - kH\sin\theta + m\pi)} \tag{126}$$

The velocity of panel A is $Ai\omega e^{i\omega t}$ and panel B is $Bi\omega e^{i\omega t}$. The mechanical impedances are:

$$Z_{m_A} = \frac{F_{A_{\text{Total}}}}{Ai\omega e^{i\omega t}} = \frac{-4iP}{Ak\omega\sin\theta}\sin\left(\frac{kH\sin\theta}{2}\right) + \frac{2ik\rho_0 c}{H}\sum_{m=-\infty}^{\infty}\frac{1}{k_{x_m}}\left(\frac{m\pi}{H} - k\sin\theta\right)^{-2}\sin\left(\frac{m\pi}{2} - \frac{kH\sin\theta}{2}\right)$$
$$\left[\left(e^{-ik\sin\theta\frac{H}{2}} - \frac{B}{A}e^{ik\sin\theta\frac{H}{2}}\right)(-i)^m + \left(\frac{B}{A}e^{-ik\sin\theta\frac{H}{2}} - e^{ik\sin\theta\frac{H}{2}}\right)(i)^m\right] \tag{127}$$

$$Z_{m_B} = \frac{F_{B_{\text{Total}}}}{Bi\omega e^{i\omega t}} = \frac{-4iP}{Bk\omega\sin\theta}\sin\left(\frac{kH\sin\theta}{2}\right)e^{-ikH\sin\theta} + \frac{2ik\rho_0 c}{H}\sum_{m=-\infty}^{\infty}\frac{1}{k_{x_m}}\left(\frac{m\pi}{H} - k\sin\theta\right)^{-2}\sin\left(\frac{m\pi}{2} - \frac{kH\sin\theta}{2}\right)$$
$$\left[\left(e^{-ik\sin\theta\frac{H}{2}} - \frac{B}{A}e^{ik\sin\theta\frac{H}{2}}\right)(-i)^m + \left(\frac{B}{A}e^{-ik\sin\theta\frac{H}{2}} - e^{ik\sin\theta\frac{H}{2}}\right)(i)^m\right]e^{-i(kH\sin\theta - m\pi)} \tag{128}$$

Eqns 127 and 128 are coupled functions of *A* and *B*. Solving for these displacement amplitudes yields:

$$A = \frac{a_1b_3 - a_2b_1 - a_1Z_B}{a_2b_2 - a_3b_3 + b_3Z_A + a_3Z_B - Z_AZ_B} \tag{129}$$

$$B = \frac{b_1a_3 - b_2a_1 - b_1Z_A}{a_2b_2 - a_3b_3 + b_3Z_A + a_3Z_B - Z_AZ_B} \tag{130}$$

$$a_1 = \frac{-4iP}{k\omega\sin\theta}\sin\left(\frac{kH\sin\theta}{2}\right) \tag{131}$$

$$b_1 = \frac{-4iP}{k\omega\sin\theta}\sin\left(\frac{kH\sin\theta}{2}\right)e^{-ikH\sin\theta} \tag{132}$$

$$a_2 = \frac{2ik\rho_0 c}{H}\sum_{m=-\infty}^{\infty}\frac{1}{k_{x_m}}\left(\frac{m\pi}{H} - k\sin\theta\right)^{-2}\sin\left(\frac{m\pi}{2} - \frac{kH\sin\theta}{2}\right)\left[(i)^m e^{-ik\sin\theta\frac{H}{2}} - (-i)^m e^{ik\sin\theta\frac{H}{2}}\right] \tag{133}$$

$$b_2 = \frac{2ik\rho_0 c}{H}\sum_{m=-\infty}^{\infty}\frac{1}{k_{x_m}}\left(\frac{m\pi}{H} - k\sin\theta\right)^{-2}\sin\left(\frac{m\pi}{2} - \frac{kH\sin\theta}{2}\right)e^{-i(kH\sin\theta - m\pi)}\left[(-i)^m e^{-ik\sin\theta\frac{H}{2}} - (i)^m e^{ik\sin\theta\frac{H}{2}}\right] \tag{134}$$

$$a_3 = \frac{2ik\rho_0 c}{H}\sum_{m=-\infty}^{\infty}\frac{1}{k_{x_m}}\left(\frac{m\pi}{H} - k\sin\theta\right)^{-2}\sin\left(\frac{m\pi}{2} - \frac{kH\sin\theta}{2}\right)\left[(-i)^m e^{-ik\sin\theta\frac{H}{2}} - (i)^m e^{ik\sin\theta\frac{H}{2}}\right] \tag{135}$$

$$b_3 = \frac{2ik\rho_0 c}{H}\sum_{m=-\infty}^{\infty}\frac{1}{k_{x_m}}\left(\frac{m\pi}{H} - k\sin\theta\right)^{-2}\sin\left(\frac{m\pi}{2} - \frac{kH\sin\theta}{2}\right)e^{-i(kH\sin\theta - m\pi)}\left[(i)^m\, e^{-ik\sin\theta\frac{H}{2}} - (-i)^m\, e^{ik\sin\theta\frac{H}{2}}\right] \tag{136}$$

Nondimensionalized Pressure and Velocity:

Pressure is nondimensionalized by the incident amplitude P and velocity is nondimensionalized by $\frac{P}{\rho_0 c}$. This requires first nondimensionalizing the displacement amplitudes A and B by $\frac{\omega\rho_0 c}{P}$. The result is:

$$\bar{A} = \frac{\bar{a}_1\bar{b}_3 - \bar{a}_2\bar{b}_1 - \bar{a}_1\bar{Z}_B}{\bar{a}_2\bar{b}_2 - \bar{a}_3\bar{b}_3 + \bar{b}_3\bar{Z}_A + \bar{a}_3\bar{Z}_B - \bar{Z}_A\bar{Z}_B} \tag{137}$$

$$\bar{B} = \frac{\bar{b}_1\bar{a}_3 - \bar{b}_2\bar{a}_1 - \bar{b}_1\bar{Z}_A}{\bar{a}_2\bar{b}_2 - \bar{a}_3\bar{b}_3 + \bar{b}_3\bar{Z}_A + \bar{a}_3\bar{Z}_B - \bar{Z}_A\bar{Z}_B} \tag{138}$$

$$\bar{a}_1 = \frac{-4i}{k_N\bar{\omega}\sin\theta}\sin\left(\frac{k_N\bar{\omega}\sin\theta}{2}\right) \tag{139}$$

$$\bar{b}_1 = \frac{-4i}{k_N\bar{\omega}\sin\theta}\sin\left(\frac{k_N\bar{\omega}\sin\theta}{2}\right)e^{-ik_N\bar{\omega}\sin\theta} \tag{140}$$

$$\bar{a}_2 = 2i\sum_{m=-\infty}^{\infty}\frac{k_N\bar{\omega}}{k_{x_m}}(m\pi - k_N\bar{\omega}\sin\theta)^{-2}\sin\left(\frac{m\pi}{2} - \frac{k_N\bar{\omega}\sin\theta}{2}\right)\left[(i)^m\, e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - (-i)^m\, e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right] \tag{141}$$

$$\bar{b}_2 = 2i\sum_{m=-\infty}^{\infty}\frac{k_N\bar{\omega}}{k_{x_m}}(m\pi - k_N\bar{\omega}\sin\theta)^{-2}\sin\left(\frac{m\pi}{2} - \frac{k_N\bar{\omega}\sin\theta}{2}\right)e^{-i(k_N\bar{\omega}\sin\theta - m\pi)}\left[(-i)^m\, e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - (i)^m\, e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right] \tag{142}$$

$$\bar{a}_3 = 2i\sum_{m=-\infty}^{\infty}\frac{k_N\bar{\omega}}{k_{x_m}}(m\pi - k_N\bar{\omega}\sin\theta)^{-2}\sin\left(\frac{m\pi}{2} - \frac{k_N\bar{\omega}\sin\theta}{2}\right)\left[(-i)^m\, e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - (i)^m\, e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right] \tag{143}$$

$$\bar{b}_3 = 2i\sum_{m=-\infty}^{\infty}\frac{k_N\bar{\omega}}{k_{x_m}}(m\pi - k_N\bar{\omega}\sin\theta)^{-2}\sin\left(\frac{m\pi}{2} - \frac{k_N\bar{\omega}\sin\theta}{2}\right)e^{-i(k_N\bar{\omega}\sin\theta - m\pi)}\left[(i)^m\, e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - (-i)^m\, e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right] \tag{144}$$

$$\bar{p}_{AB}(\bar{x},\bar{y},\bar{t}) = \frac{k_N\bar{\omega}}{2}\sum_{n=-\infty}^{\infty}\frac{1}{\bar{k}_{x_n}}(n\pi - k_N\bar{\omega}\sin\theta)^{-1}\left[\left(\bar{A}e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - \bar{B}e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right)(-i)^n + \left(\bar{B}e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - \bar{A}e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right)(i)^n\right] e^{i\left(\bar{\omega}\bar{t} - \bar{k}_{x_n}\bar{x} - (k_N\bar{\omega}\sin\theta - n\pi)\bar{y}\right)} \tag{145}$$

$$\bar{U}_{AB}(\bar{x},\bar{y},\bar{t}) = \frac{1}{2}\sum_{q=-\infty}^{\infty}(q\pi - k_N\bar{\omega}\sin\theta)^{-1}\left[\left(\bar{A}e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - \bar{B}e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right)(-i)^q + \left(\bar{B}e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - \bar{A}e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right)(i)^q\right] e^{i\left(\bar{\omega}\bar{t} - \bar{k}_{x_q}\bar{x} - (k_N\bar{\omega}\sin\theta - q\pi)\bar{y}\right)} \tag{146}$$

$$\bar{Z}_A = \frac{1}{\zeta_{aca}}\left[2\zeta_a + i\bar{\omega}\left(1 - \frac{1}{\bar{\omega}^2}\right)\right], \bar{Z}_B = \frac{1}{\zeta_{aca}}\left[2\zeta_a + i\bar{\omega}\left(1 - \frac{1}{\bar{\omega}^2}\right)\right]$$
$$kH = k_N\bar{\omega}, k_N = \frac{\omega_{N_A}H}{c}, \bar{\omega} = \frac{\omega}{\omega_N}, \omega_R = \frac{\omega_{N_B}}{\omega_{N_A}}, \bar{k}_{x_n} = Hk_{x_n}, \bar{k}_{y_n} = Hk_{y_n}, \bar{t} = t\omega_{N_A}, \bar{x} = \frac{x}{H}, \bar{y} = \frac{y}{H}$$

$$\bar{k}_{x_n} = \sqrt{\bar{k}^2 - \bar{k}_{y_n}^2} = \sqrt{(k_N\bar{\omega})^2 - (k_N\bar{\omega}\sin\theta - \pi n)^2} = -i\sqrt{(k_N\bar{\omega}\sin\theta - \pi n)^2 - (k_N\bar{\omega})^2} \tag{147}$$

For a mode to radiate, $\bar{k}_{x_n}$ must be real. It follows that

$$\frac{-k_N\bar{\omega}\,(1 - \sin\theta)}{\pi} < n_{\text{Radiating}} < \frac{k_N\bar{\omega}\,(1 + \sin\theta)}{\pi} \tag{148}$$

Power Radiated Downstream:

The real components of the radiating pressure and velocity, given in Eqns 98 and 99, are

$$\bar{p}_{\text{Real}_{AB}}(\bar{x}, \bar{y}, \bar{t}) = \frac{k_N\bar{\omega}}{2}\sum_{n_{\text{Rad}}}\frac{1}{\bar{k}_{x_n}}(n\pi - k_N\bar{\omega}\sin\theta)^{-1}\left|\left(\bar{A}e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - \bar{B}e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right)(-i)^n + \left(\bar{B}e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - \bar{A}e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right)(i)^n\right|$$
$$\cos\left(\bar{\omega}\bar{t} - \bar{k}_{x_n}\bar{x} - (k_N\bar{\omega}\sin\theta - n\pi)\,\bar{y} + \phi\right) \tag{149}$$

$$\bar{u}_{\text{Real}_{AB}}(\bar{x}, \bar{y}, \bar{t}) = \frac{1}{2}\sum_{q_{\text{Rad}}}(q\pi - k_N\bar{\omega}\sin\theta)^{-1}\left|\left(\bar{A}e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - \bar{B}e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right)(-i)^q + \left(\bar{B}e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - \bar{A}e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right)(i)^q\right|$$
$$\cos\left(\bar{\omega}\bar{t} - \bar{k}_{x_q}\bar{x} - (k_N\bar{\omega}\sin\theta - q\pi)\,\bar{y} + \phi\right) \tag{150}$$

The power radiated downstream is the product of the real components of the radiating pressure and velocity:

$$\begin{aligned}\Pi_{AB}(\bar{x}, \bar{y}, \bar{t}) = \frac{k_N\bar{\omega}}{8}\sum_{n_{\text{Rad}}}\sum_{q_{\text{Rad}}}\frac{1}{\bar{k}_{x_n}}(n\pi - k_N\bar{\omega}\sin\theta)^{-1}(q\pi - k_N\bar{\omega}\sin\theta)^{-1}\\ \left|\left(\bar{A}e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - \bar{B}e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right)(-i)^n + \left(\bar{B}e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - \bar{A}e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right)(i)^n\right|\\ \left|\left(\bar{A}e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - \bar{B}e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right)(-i)^q + \left(\bar{B}e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - \bar{A}e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right)(i)^q\right|\\ \cos\left(\bar{\omega}\bar{t} - \bar{k}_{x_n}\bar{x} - (k_N\bar{\omega}\sin\theta - n\pi)\,\bar{y} + \phi\right)\cos\left(\bar{\omega}\bar{t} - \bar{k}_{x_q}\bar{x} - (k_N\bar{\omega}\sin\theta - q\pi)\,\bar{y} + \phi\right)\end{aligned} \tag{151}$$

After averaging in time:

$$\begin{aligned}\langle\Pi_{AB}\rangle_{\bar{t}} = \frac{k_N\bar{\omega}}{8}\sum_{n_{\text{Rad}}}\sum_{q_{\text{Rad}}}\frac{1}{\bar{k}_{x_n}}(n\pi - k_N\bar{\omega}\sin\theta)^{-1}(q\pi - k_N\bar{\omega}\sin\theta)^{-1}\\ \left|\left(\bar{A}e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - \bar{B}e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right)(-i)^n + \left(\bar{B}e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - \bar{A}e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right)(i)^n\right|\\ \left|\left(\bar{A}e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - \bar{B}e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right)(-i)^q + \left(\bar{B}e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - \bar{A}e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right)(i)^q\right|\\ \cos\left(\left(\bar{k}_{x_q} - \bar{k}_{x_n}\right)\bar{x} + \pi\,(n - q)\,\bar{y}\right)\end{aligned} \tag{152}$$

After the spatial average in $\bar{y}$, the average radiated pressure downstream is:

$$\langle\langle\Pi_{AB}\rangle_{\bar{t}}\rangle_{\bar{y}} = \frac{k_N\bar{\omega}}{4}\sum_{n_{\text{Rad}}}\frac{1}{\bar{k}_{x_n}}(n\pi - k_N\bar{\omega}\sin\theta)^{-2}\left|\left(\bar{A}e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - \bar{B}e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right)(-i)^n + \left(\bar{B}e^{\frac{-ik_N\bar{\omega}\sin\theta}{2}} - \bar{A}e^{\frac{ik_N\bar{\omega}\sin\theta}{2}}\right)(i)^n\right|^2 \tag{153}$$

$$\bar{k}_{x_n} = -i\sqrt{(k_N\bar{\omega}\sin\theta - \pi n)^2 - (k_N\bar{\omega})^2}$$

$$\frac{-k_N\bar{\omega}(1-\sin\theta)}{\pi} < n_{\text{Rad}_{AB}} < \frac{k_N\bar{\omega}(1+\sin\theta)}{\pi}$$

4 Results

Since the goal of this project is to investigate the sound blocking abilities of a barrier, the primary results of interest are plots of the average power radiating in the far-field (where a finite number of modes radiate). Plotting power vs. the nondimensionalized angular frequency $\bar{\omega}$ reveals at which source frequencies ω, relative to the natural frequency of panel A ω_{N_A}, the barrier is the most effective at reducing sound transmission. Figure 9 is an example of such a graph, where the power radiated from two panels in a duct (Eqn 59) is plotted vs $\bar{\omega}$. This plot demonstrates important features shared by all of the power plots included in this paper.

The y-axis is scaled from 0 to 1 because during the calculation of power, terms such as pressure and velocity were nondimensionalized by the pressure and velocity amplitudes of the incident wave, respectively. Therefore, 0 power indicates that there is complete cancellation of power at that frequency and 1 indicates that the panels are perfectly transparent and there is perfect transmission.

The panels are acoustically transparent at their natural frequencies, allowing perfect transmission. At mechanical resonance, $\omega = \omega_{N_A}$ and therefore, $\bar{\omega} = 1$. According to the nondimensionalized expression for impedance in Eqn 13, when there is no structural damping, $\bar{Z} = 0$, and it is as if the panel barrier is not there, allowing perfect transmission. Because of how $\bar{\omega}$ has been defined, resonance *always* occurs at $\bar{\omega} = 1$ when $\omega = \omega_{N_A}$, resulting in the right peak in Figure 9. The left peak can be shifted based on the natural frequency of the B panel: resonance with respect to the B panel occurs at $\omega = \omega_R$ which in this plot is equal to 0.5. There is perfect cancellation of radiated power at the driving frequency between the two panels' natural frequencies because that is when the panels are the most out of phase and the "sloshing" or backloading effect is maximized.

4.1 Two Panels in a Duct

The expression for average radiated power for two panels in a duct from Eqn 59 is a function of the fluid loading of both panels, ζ_{ac_a} and ζ_{ac_b}, the structural damping, ζ_a and ζ_b, the natural frequency of the B panel, ω_R, the duct height relative to the source wavelength (determined through $\bar{k}$), and the frequency at which higher modes begin to cut on, $\bar{\omega}_{c/o}$. The objective of this project is to minimize the radiating power across the most frequencies, so it will now be investigated how changing these different parameters can favorably change the shape of Figure 9.

Nondimensionalized impedances $\bar{Z}_a$ and $\bar{Z}_b$ are inversely proportional to the fluid loadings or acoustic damping terms ζ_{ac_a} and ζ_{ac_b} as shown in Eqn 13. Therefore, as ζ_{ac_a} and ζ_{ac_b} increase, $\bar{Z}_a$ and $\bar{Z}_b$ decrease and approach zero, or the case of acoustic transparency. Therefore, as the fluid loading is increased, the frequency band at which the power radiated is close to 1 broadens. This effect is shown in Figure 10 as both ζ_{ac_a} and ζ_{ac_b} increase from 0.1 to 2 to 5. Therefore for the purpose of reducing sound transmission,

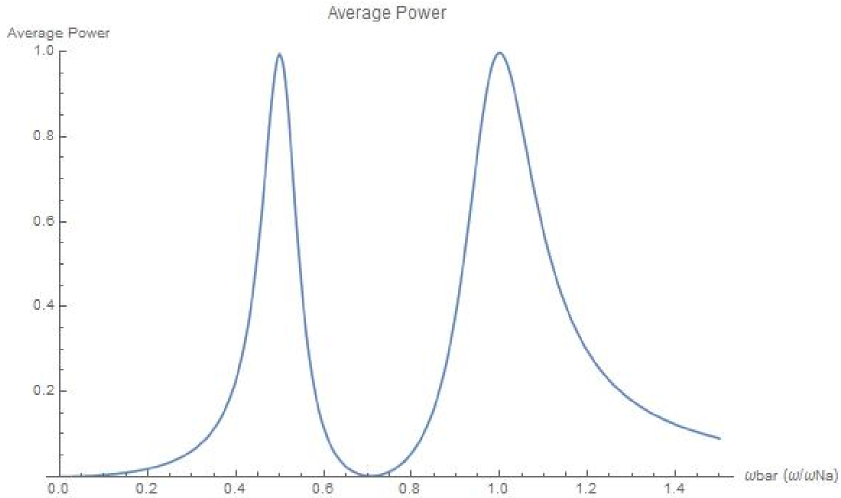

Figure 9: The average power radiating downstream from two panels in a duct is plotted vs $\bar{\omega}$. The two peaks indicate resonance where the panels are acoustically transparent becuase the system is being driven at the natural frequency of one of the panels.

desirable mass-spring-dampers will have low acoustic damping. Note that the "glitches" in this plot where it appears that the power goes to infinity at various $\bar{\omega}$ indicate where higher modes are turning on, but the power always remains finite and less than one (In making this plot, Mathematica finds that the power is indeterminate when $\bar{k}_{x_m} = 0$ and is in a denominator. In actuality, there is always a $\bar{k}_{x_m} = 0$ term in the numerator that adequately cancels this term out and the power does not go to infinity).

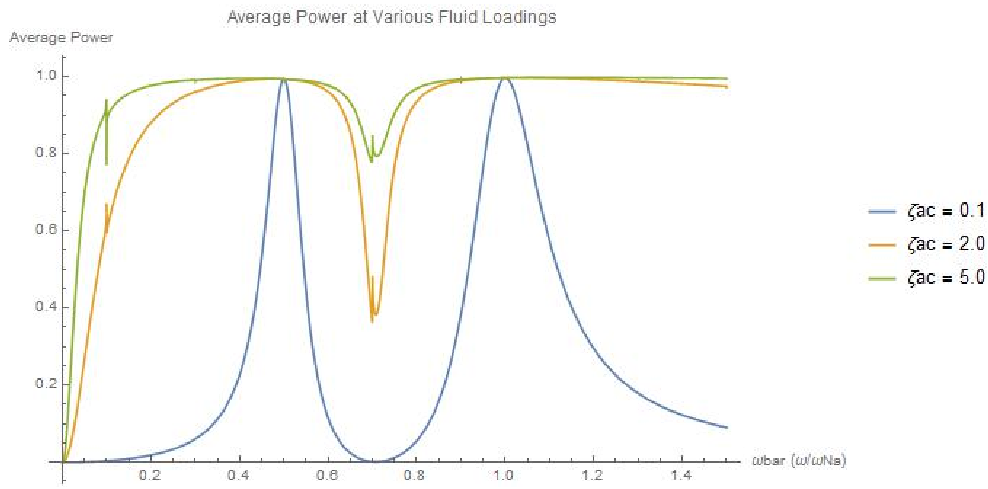

Figure 10: The average power radiating downstream from two panels in a duct is plotted vs $\bar{\omega}$ for various fluid loadings. As the acoustic damping is increased, the band at which the panels are effectively acoustically transparent broadens.

Conversely, increasing the structural damping ζ_a and ζ_b values increases the impedances of the two panels and allows the panels to block more sound. As the structural damping is increased, the resonant peaks decrease in magnitude as shown in Figure 11 as ζ_a and ζ_b both increase from 0 to 0.05 to 0.3. An ideal panel, then, would have large structural damping.

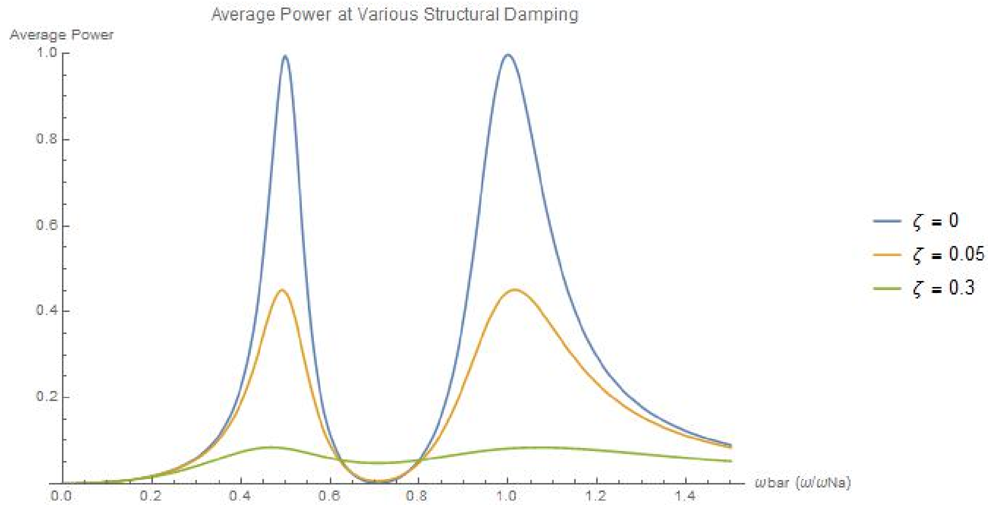

Figure 11: The average power radiating downstream from two panels in a duct is plotted vs $\bar{\omega}$ for various structural damping values. As the structural damping is increased, the resonant peaks decrease in magnitude.

As previously mentioned, the locations of the resonant peaks are determined by the natural frequencies of the panels. Since $\bar{\omega}$ is defined to be the ratio of ω to ω_{N_A}, the resonant peak at $\bar{\omega} = 1$ which indicates that the system is driven at the natural frequency of panel A, never moves away from $\bar{\omega} = 1$. The location of the second resonant peak is determined by the natural frequency of panel B since $\omega_R = \frac{\omega_{N_B}}{\omega_{N_A}}$. Figure 12 includes plots of power given four different ω_R values: 0.5, 1, 3, and 4. As the resonant peaks become further apart, the band of frequencies between 1 and ω_R where sound cancellation is most effective widens. The greatest sound cancellation can be achieved when the two panels are tuned to drastically different natural frequencies.

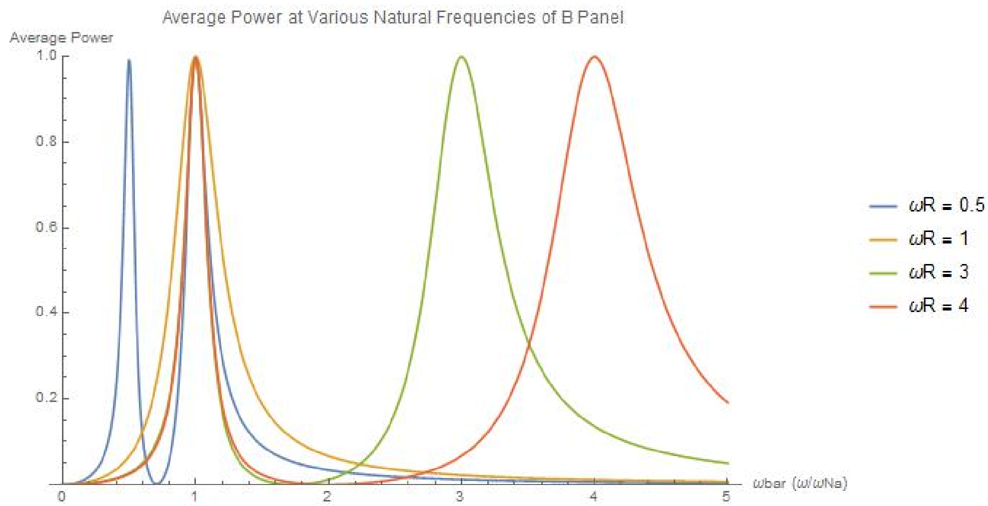

Figure 12: The average power radiating downstream from two panels in a duct is plotted vs $\bar{\omega}$ for various ratios of the natural frequency of panel B to A. The resonant peak at $\bar{\omega} = 1$ indicates that the system is driven at the natural frequency of panel A. Given how $\bar{\omega}$ is defined, this peak never moves. If the natural frequency of panel B is lower or higher than that of panel A, the other resonant peak may stand to the left or right of this peak at $\bar{\omega} = 1$. The further apart the resonant peaks are, the greater the sound cancellation is in between.

Looking back at the power expression in Eqn 59, the radiated power is the sum of the $m = 0$ mode and higher modes. When the two panels are identical, $\bar{U}_a = \bar{U}_b$ and the two panels act as one larger panel. The sum of the higher modes goes to zero and all that remains is the $m = 0$ mode, which simply gives the power of the one panel one-dimensional case as we expected. If, for two panels, the average power is heavily dominated by the $m = 0$ mode, then the system closely resembles the one-dimensional case. This occurs when the duct height is much smaller than the wavelength λ of the sound source. The parameter $\bar{k}$ is defined as follows: $\bar{k} = kH = \frac{2\pi H}{\lambda}$. As $\bar{k}$ increases, the duct height increases relative λ so we expect higher modes to contribute more significantly to the total power. This effect is seen in the comparison of Figures 13 and 14. These plots include both the total average power (orange) and the contribution to power from just the $m = 0$ mode (blue). The latter is not in view in Figure 13 because it is identical to the total power: when $\bar{k} = 0$, no higher modes contribute to the power. This can be seen from the expression for power ($\bar{k} = 0$ is in the numerator of the higher mode sum) and we expect this result since we treat the system as one-dimensional when $\bar{k} = 0$. In Figure 14, conversely, $\bar{k} = 5$ and now the higher modes contribute in a significant way as shown by the discrepancy between the total average power and the contribution from just the $m = 0$ mode. In addition, the amplitudes of the resonant peaks decrease to less than 1. Increasing the relative duct height reduces transmission because the piecewise or "stair-step" discontinuity between the two panels cannot be "smoothed over" by a large wavelength. Larger $\bar{k}$ values are therefore favorable for purposes of reducing sound transmission.

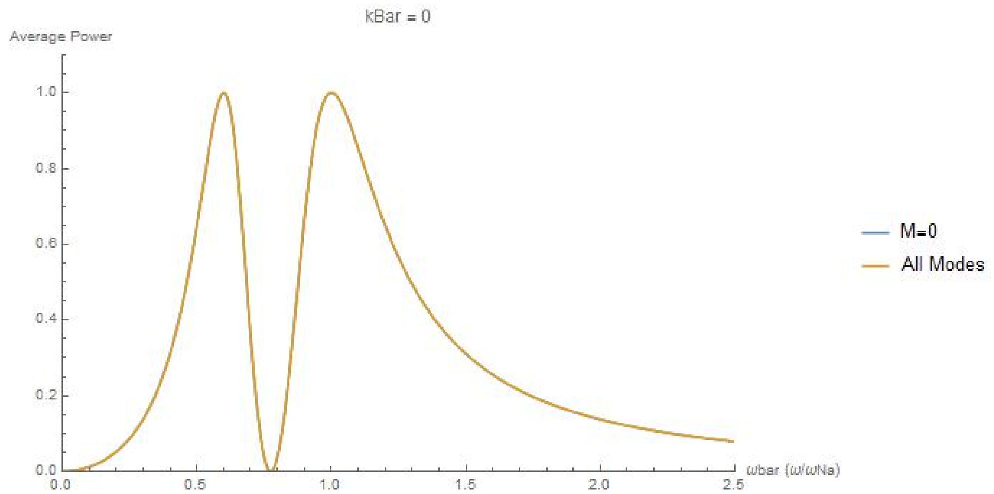

Figure 13: The average power radiating downstream from two panels in a duct is plotted vs $\bar{\omega}$ for $\bar{k} = 0$. In this limit, there are no higher modes contributing to the average power.

Recall from Eqn 52 that $\bar{\omega}_{c/o}$ is the frequency at which the first higher mode turns on (beause for $\bar{\omega} < \bar{\omega}_{c/o}$, $\bar{k}_{x_m}$ is imaginary and no higher modes radiate). For $\bar{\omega} < \bar{\omega}_{c/o}$, the total power is composed then of only the $m = 0$ mode. As a result, if $\bar{\omega}_{c/o}$ is much larger than ω_R and 1 such that higher modes do not turn on until $\bar{\omega}$ is large and the resonant peaks have already appeared, then the $m = 0$ mode is a good approximation for the total power. This is shown in Figure 15 in which the total power and the power from the $m = 0$ mode do not deviate until after the second resonant peak since $\bar{\omega}_{c/o}$ is 1.5.

As $\bar{\omega}_{c/o}$ decreases, however, the $m = 0$ mode becomes a worse approximation since higher modes turn on sooner ($\bar{\omega}$-wise). Higher modes cannot be ignored, especially when a higher mode turns on near a resonant peak where the power has the most "weight". Figure 16 shows how the smooth, rounded resonant peak at $\bar{\omega} = 1$ has been disrupted by the first higher mode turning on at $\bar{\omega} = 0.75$

When $\bar{\omega}_{c/o}$ is less than both ω_R and 1, higher modes significantly affect the total radiated power at

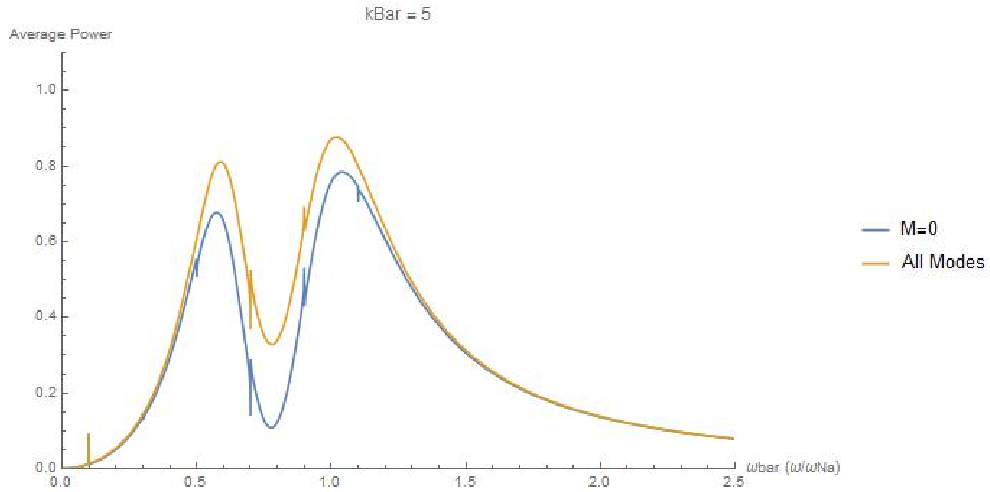

Figure 14: The average power radiating downstream from two panels in a duct is plotted vs $\bar{\omega}$ for $\bar{k} = 5$. In this limit, higher modes contribute significantly to the total average power, so the discrepancy between the total power and just the $m = 0$ contribution is more drastic. In addition, the resonant peaks have decreased in magnitude.

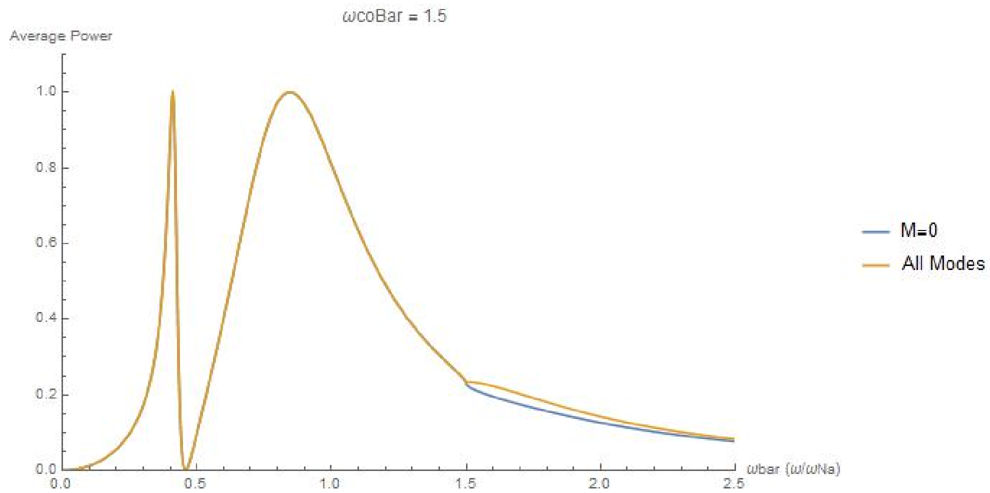

Figure 15: The average power radiating downstream from two panels in a duct is plotted vs $\bar{\omega}$ for $\bar{\omega}_{c/o} = 1.5$. No higher modes turn on until $\bar{\omega} = 1.5$, so the $m = 0$ mode is an excellent approximation for the total average radiated power.

both resonant peaks: the $m = 0$ mode alone is not a good approximation for the total power as seen by the discrepancy between the two plots in Figure 17. In addition, the "disruption" created by the higher modes results in lower amplitudes of the resonant peaks. Therefore turning on higher modes sooner (making $\bar{\omega}_{c/o}$ small) yields lower sound transmission.

In conclusion, to reduce sound transmission downstream from a duct with two panels, one should choose a set-up with low acoustic damping, large structural damping, large duct height relative to source wavelength, and $\bar{\omega}_{c/o}$ less than both ω_R and 1. The choice of how to tune the natural frequency of the B panel depends on the frequencies of the sound source that you wish to cancel. If the system is driven at low $\bar{\omega}$, choose a large ω_R and vice versa.

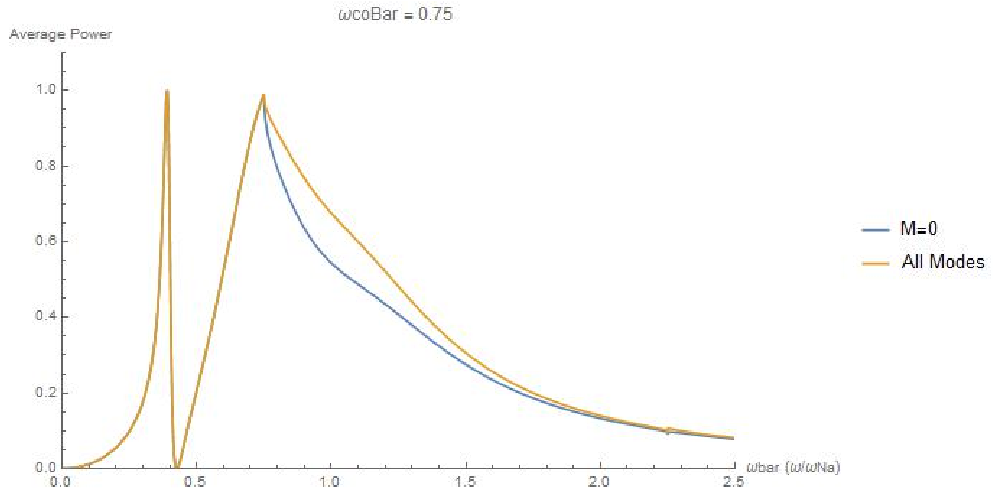

Figure 16: The average power radiating downstream from two panels in a duct is plotted vs $\bar{\omega}$ for $\bar{\omega}_{c/o} = 0.75$. Turning on a higher mode near a resonant peak means that the shape of the peak is disrupted and the $m = 0$ mode alone is not a good approximation for the total radiated power.

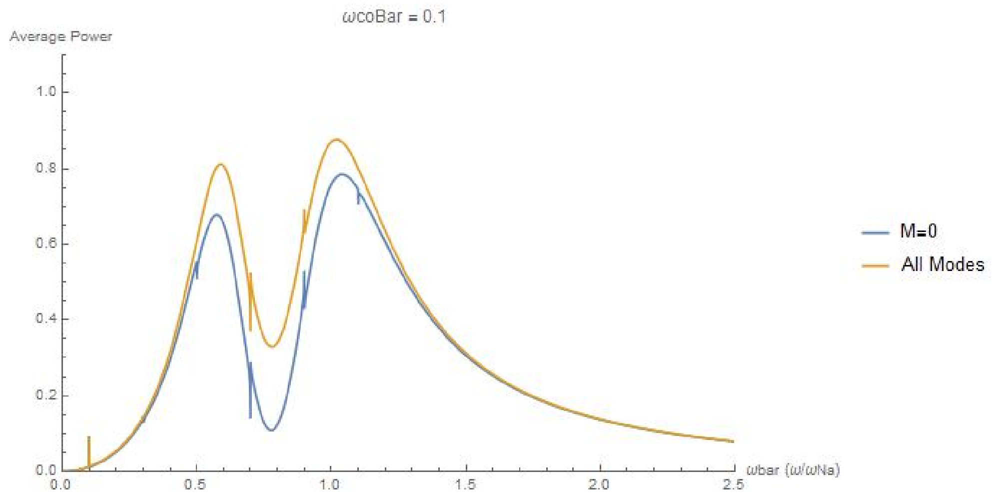

Figure 17: The average power radiating downstream from two panels in a duct is plotted vs $\bar{\omega}$ for $\bar{\omega}_{c/o} = 0.1$. Turning on a higher mode near a resonant peak means that the shape of the peak is disrupted and the $m = 0$ mode alone is not a good approximation for the total radiated power.

4.2 Infinite Stack of Panels

4.2.1 Displacement

During the analysis of the infinite stack of panels, it was important to check that the complex Fourier series representation of the displacement of the boundary was correct before using it to determine the acoustic pressure, velocity, and power. Animations of the oscillating panels were created and screenshots of those animations are included below. In the calculation, two different cases are addressed: one in which the wavelength λ_y is an integer multiple of the panel height H, and the generalized case in which there is no constraint on panel height. The plots of the position y along the barrier vs the barrier's horizontal displacement are virtually indistinguishable between these two cases, so I have only included plots for the most generalized case.

When the panels are all identical, the displacement is that given in Eqn 82. If this were an animation, each panel (vertical line) would be oscillating left and right in the direction of the x-axis with a maximum displacement amplitude of 0.5.

I have included two plots of this displacement with different panel heights: Figure 18 has a panel height that yields 11.1 panel lengths per λ_y and Figure 19 yields 51.1 (and thus has a smaller panel height H). As the panel height decreases, the plot of the displacement become smoother. These plots accurately depict the barrier's displacement and match the analytical model which expressed the displacement as a sum of heaviside functions.

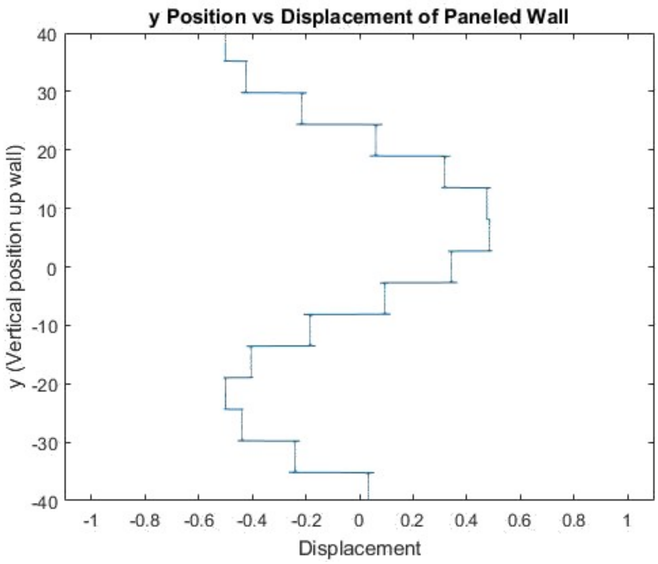

Figure 18: This plot of panel displacement is a view of the x-y plane as defined in Figure 1. If this were an animation, the panels would be oscillating to the left and right.

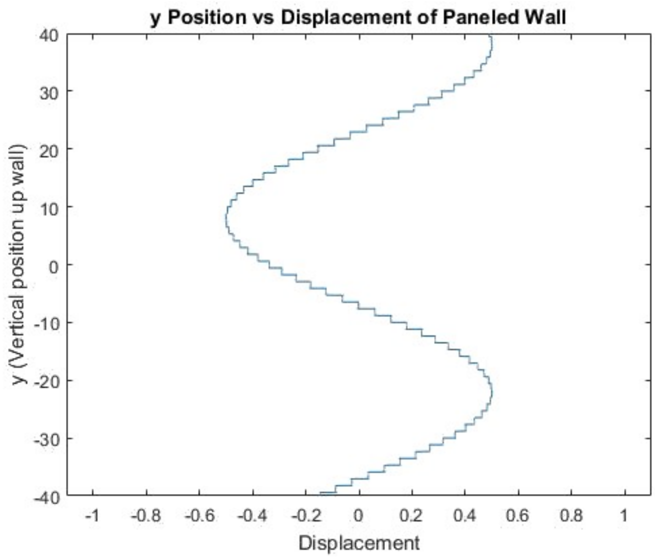

Figure 19: This plot of panel displacement is a view of the x-y plane as defined in Figure 1. As panel height decreases, the displacement curve becomes smoother and looks increasingly like a simple sinusoidal boundary.

Now if the boundary alternates between A and B panels, the expression for displacement is given in Eqn 113. The maximum displacement of panel A is chosen to be 0.5, and is 1 for panel B. In Figure 20, 11.1 panels "fit" within λ_y and it is difficult to see the wave pattern created. In Figure 21, however, the height of each panel is decreased and 51.1 panels "fit" within λ_y and the pattern becomes more visible: Both panel types oscillate such that two waves with the same wavelength, but different amplitude, overlap. This pattern is highlighted in Figure 22 where the orange line traces the motion of the A panel (amplitude = 0.5) and the green line outlines the oscillating motion of the B panel (amplitude = 1).

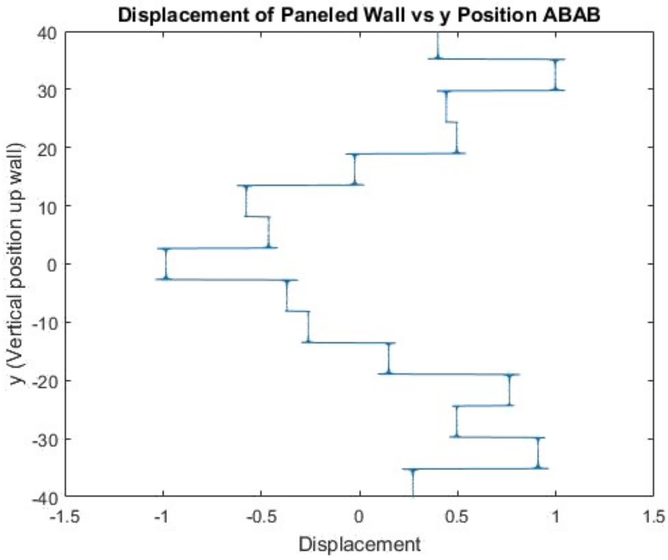

Figure 20: This plot of panel displacement is a view of the x-y plane as defined in Figure 1. If this were an animation, the panels would be oscillating to the left and right. Panel B reaches a maximum displacement amplitude of 1 while panel A's maximum displacement is 0.5.

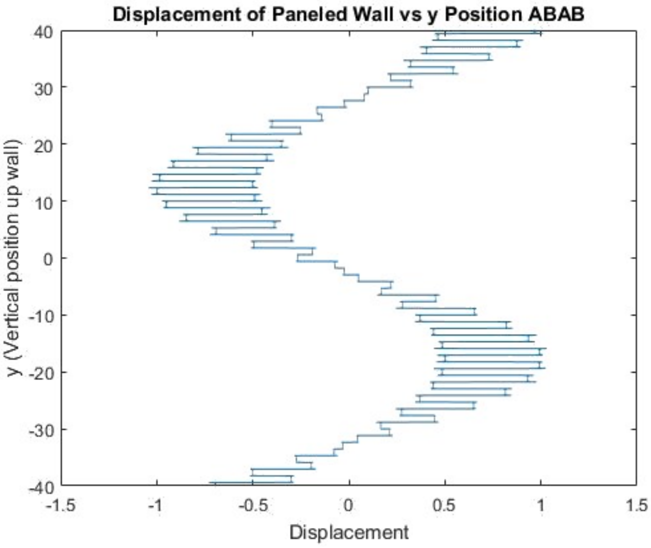

Figure 21: This plot of panel displacement is a view of the x-y plane as defined in Figure 1. As panel height decreases, the displacement curve becomes smoother and looks increasingly like a simple sinusoidal boundary. Panel B reaches a maximum displacement amplitude of 1 while panel A's maximum displacement is 0.5.

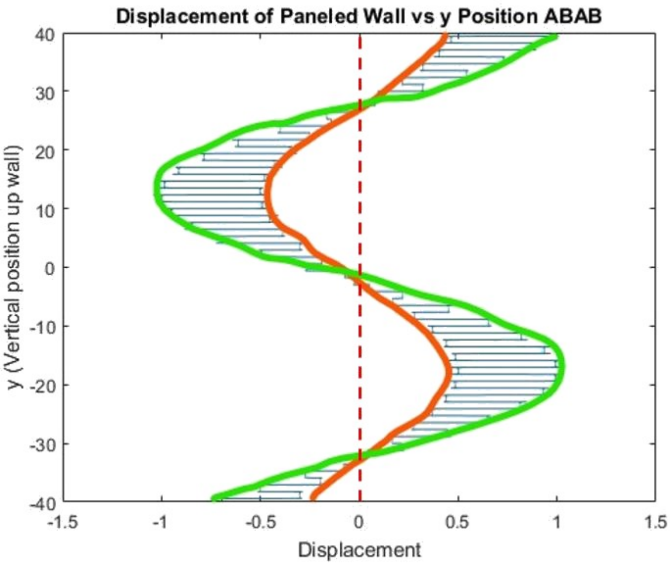

Figure 22: When every other panel is identical, there is an overlapping wave pattern in which both sets of A and B panels oscillate with the same wavelength but at different amplitudes.

4.2.2 Power

For an infinite stack of identical panels, Eqn 106 gives the expression for power radiated far away from the panels. This expression is a function of fluid loading ζ_{ac}, structural damping ζ, incident angle θ (where 0 radians indicates normal incidence), panel height as encompassed by k_N, and the ratio of the driving frequency to the natural frequency of the identical panels, $\bar{\omega}$. Because there is only one panel type, there is only one resonant peak when plotting average power vs. $\bar{\omega}$. It is of interest to see how the transmitted power can be reduced by changing ζ_{ac}, ζ, θ, and k_N.

Increasing the fluid loading or structural damping has the same effect as it did in the case of two panels in a duct. These results are shown in Figures 23 and 24, respectively.

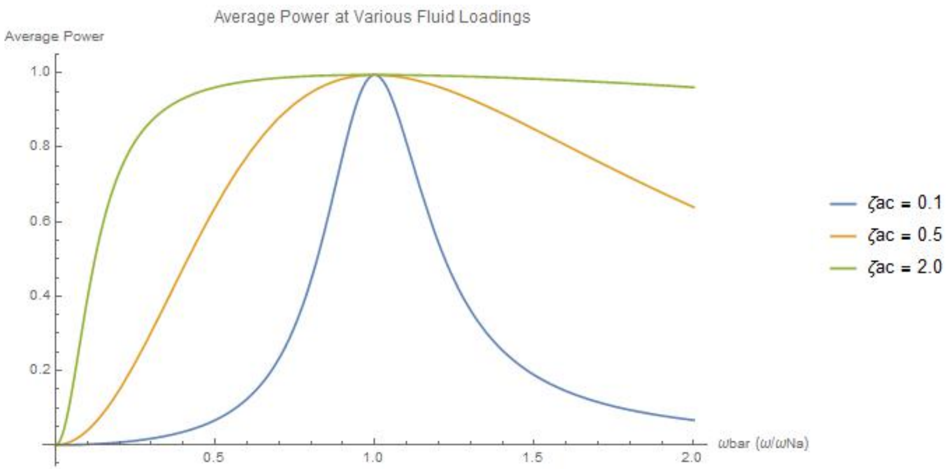

Figure 23: The average power radiating downstream from the infinitely tall stack of *identical* panels is plotted vs $\bar{\omega}$ for various fluid loadings. As the acoustic damping is increased, the band at which the panels are effectively acoustically transparent broadens.

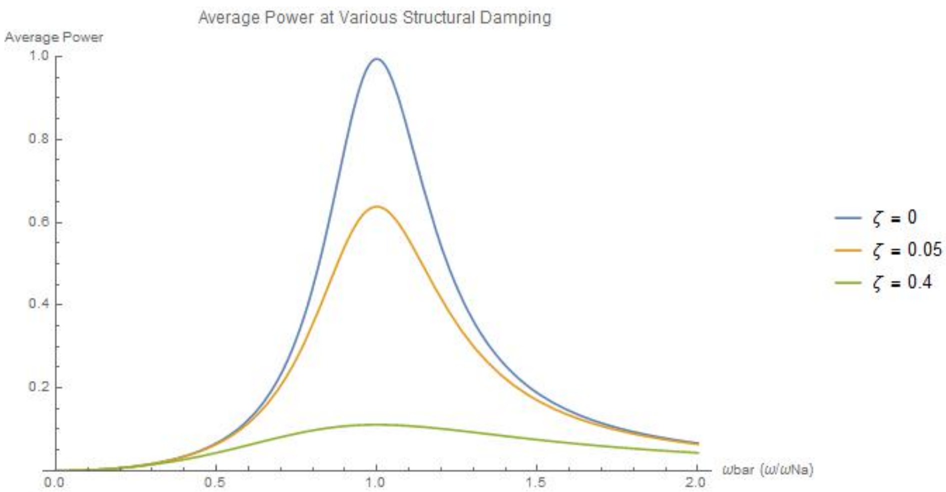

Figure 24: The average power radiating downstream from the infinitely tall stack of *identical* panels is plotted vs $\bar{\omega}$ for various structural damping values. As the structural damping is increased, the resonant peak decreases in magnitude.

The incident angle θ is measured from the normal to the panels, so that $\theta = 0$ indicates normal incidence and $\theta_{\text{Max}} = \frac{\pi}{2}$. As θ increases and the sound source's incidence becomes more oblique, the energy reflected off the panels increases so less power is transmitted. This effect is shown in Figure 25 for three incidence angles measured in radians: 0, 0.75, and 1.25.

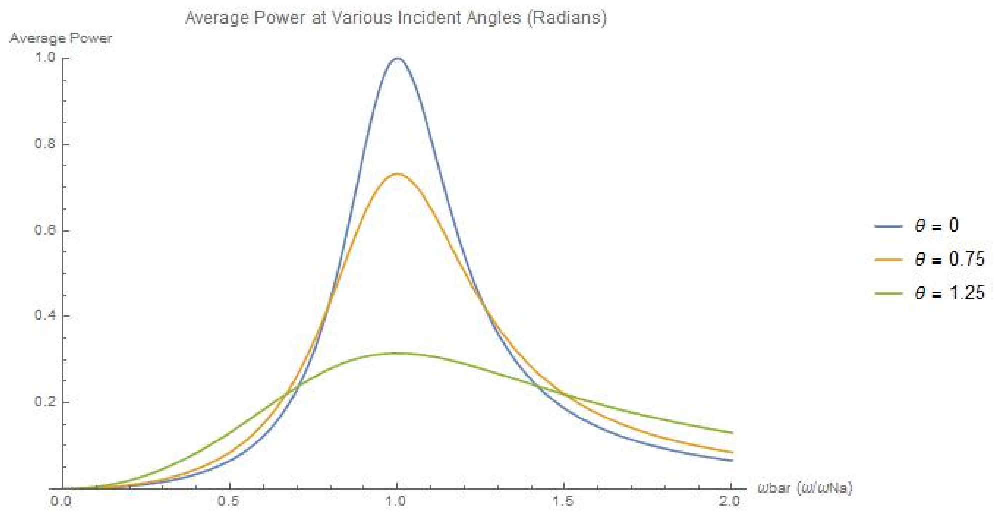

Figure 25: The average power radiating downstream from the infinitely tall stack of *identical* panels is plotted vs $\bar{\omega}$ for various incident angles. As the incidence of the sound source becomes increasingly oblique, less power is transmitted.

In the previous section, changing the term $\bar{k}$ changed the duct height relative to the wavelength of the sound source. The analogous term in this case is $k_N = \frac{\omega_N H}{c}$ which indicates the height of a single panel relative to λ_y. When k_N is very small, many panels fit within λ_y and the barrier appears to be smooth. When k_N is large, however, the stair-stepping, discontinous nature of the barrier is magnified and even at resonance, the shape of the barrier lowers transmission. It is expected, then, that increasing k_N will increase the effect of the higher modes and decrease the height of the resonant peak. This is exactly what is seen in Figure 26 in which average power is plotted at three different values of k_N: 0.1, 5, and 10.

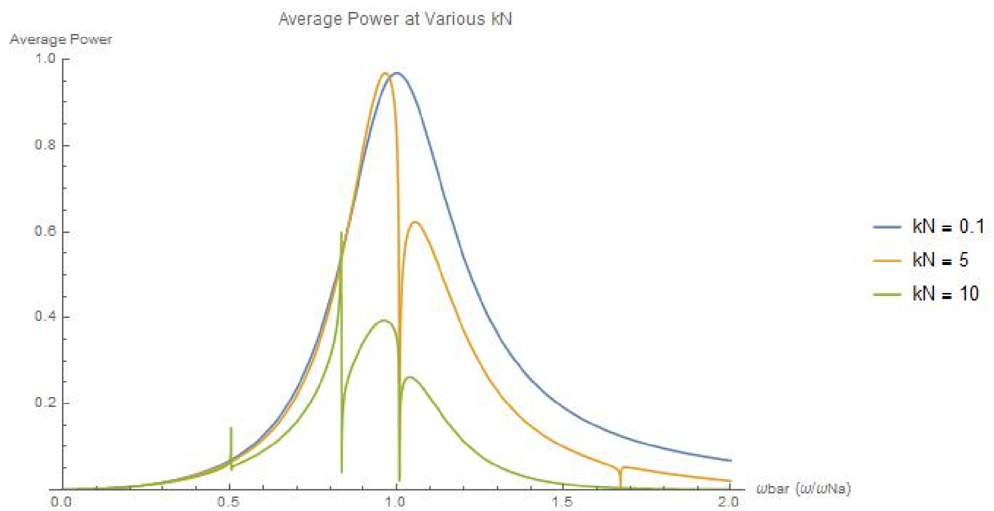

Figure 26: The average power radiating downstream from the infinitely tall stack of *identical* panels is plotted vs $\bar{\omega}$ for various panel heights as determined by k_N. As the panel height increases, the effects of higher modes become more prominent and the magnitude of the resonant peak is lowered.

For the alternating panel case when every other panel is identical, the expression for power (Eqn 153) is a function of the natural frequency of panel B (through ω_R) in addition to ζ_{ac}, ζ, θ, and k_N. Plots for average radiated power vs $\bar{\omega}$ for changing ζ_{ac}, ζ, θ, k_N, and ω_R are given below (Figures 27, 28, 29, 30, and 31). Note that, just as in the case of two panels in a duct, when a parameter such as the structural damping is changed, it is changed for both panels (the panels are not perfectly identical, however, because they always have different natural frequencies). The plots below show the expected behavior, with the exception of the plot with varying k_N values. Increasing panel height does not significantly decrease the amplitude of the resonant peaks, and instead seems to shift panel A's resonant peak away from $\bar{\omega} = 1$. This does, however, broaden the range of frequencies at which there is minimal power transmission ($\bar{\omega}$ values between the two resonant peaks), so the conclusion remains that it is advantageous to increase k_N in order to reduce sound transmission.

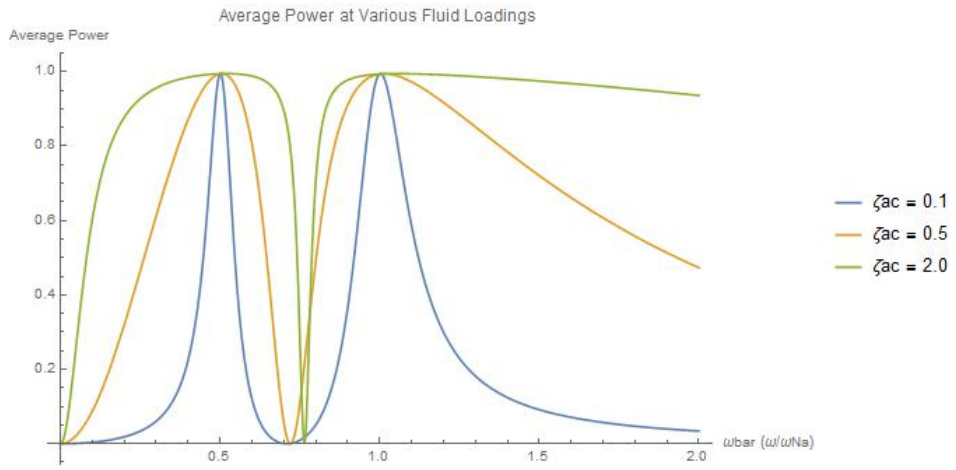

Figure 27: The average power radiating downstream from the infinitely tall stack of *alternating* panels is plotted vs $\bar{\omega}$ for various fluid loadings. As the acoustic damping is increased, the band at which the panels are effectively acoustically transparent broadens.

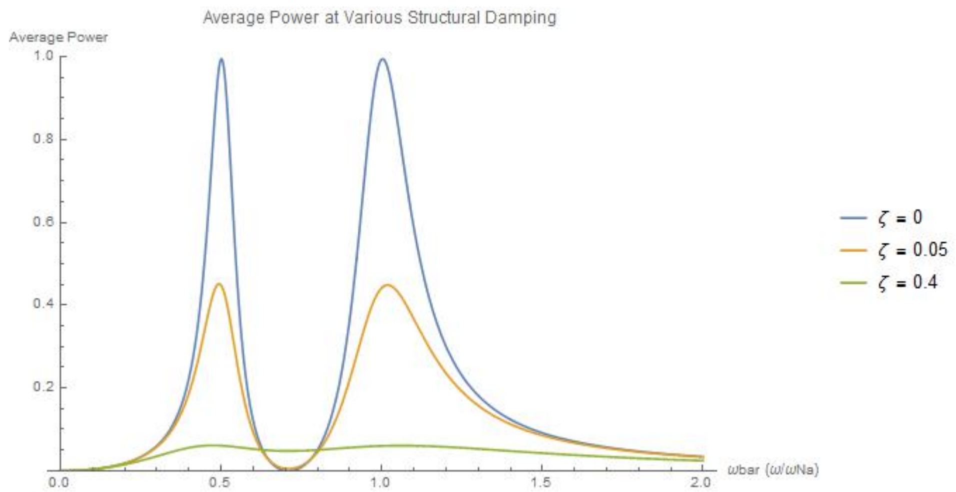

Figure 28: The average power radiating downstream from the infinitely tall stack of *alternating* panels is plotted vs $\bar{\omega}$ for various structural damping values. As the structural damping is increased, the resonant peaks decrease in magnitude.

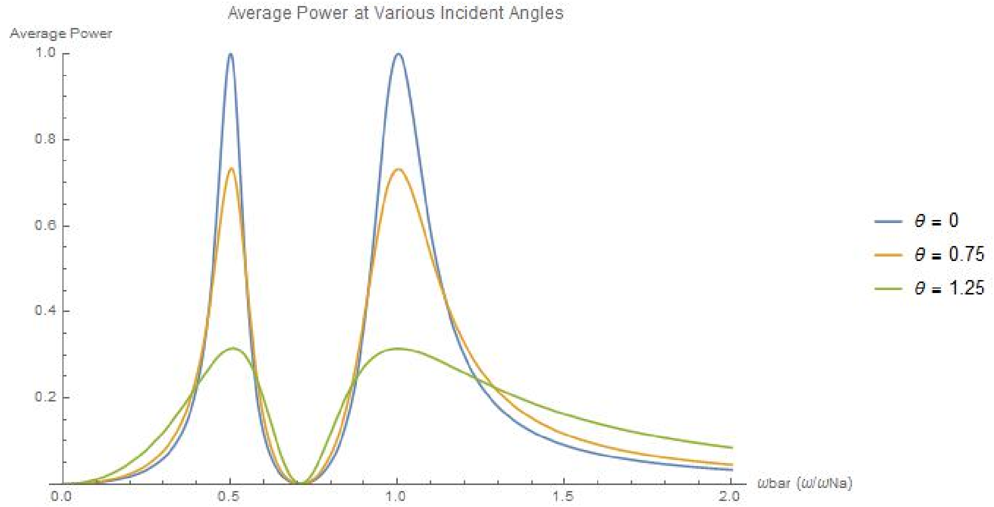

Figure 29: The average power radiating downstream from the infinitely tall stack of *alternating* panels is plotted vs $\bar{\omega}$ for various incident angles. As the incidence of the sound source becomes increasingly oblique, less power is transmitted.

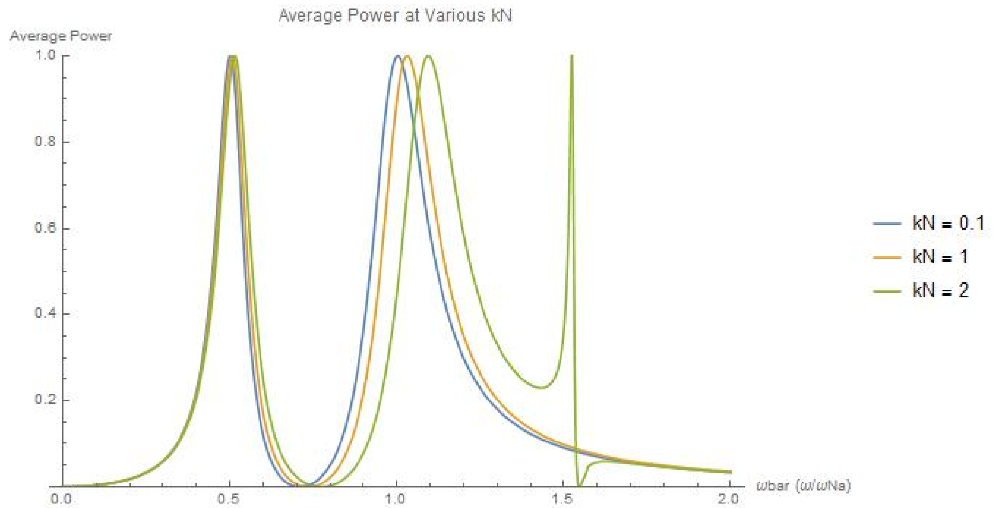

Figure 30: The average power radiating downstream from the infinitely tall stack of *alternating* panels is plotted vs $\bar{\omega}$ for various panel heights as determined by k_N. As the panel height increases, panel A's resonant peak shifts to $\bar{\omega} > 1$

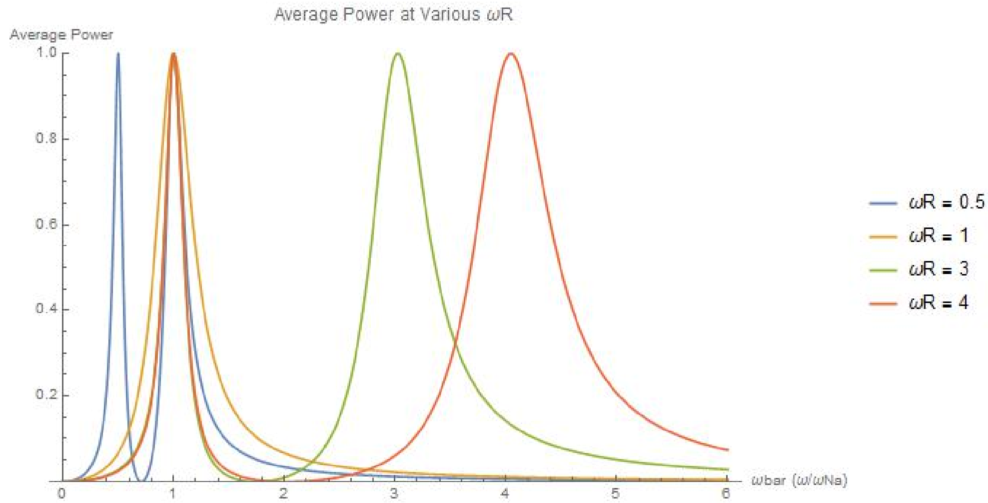

Figure 31: The average power radiating downstream from the infinitely tall stack of *alternating* panels is plotted vs $\bar{\omega}$ for various ratios of the natural frequency of panel B to A. The resonant peak at $\bar{\omega} = 1$ indicates that the system is driven at the natural frequency of panel A. If the natural frequency of panel B is lower or higher than that of panel A, the other resonant peak may stand to the left or right of this peak at $\bar{\omega} = 1$. As the resonant peaks become further apart, the band of frequencies where sound is cancelled widens.

5 Conclusion

Acoustic barriers that reduce sound transmission are relevant to several aspects of daily life, from "soundproofing" an office space to reducing the jet-engine noise in an airplane cabin. This paper proposes an acoustic barrier that is made of oscillating panels which can be given different structural properties so that they may move out of phase and cancel the sound transmitted. This set-up is reduced to two dimensions or an infinitely tall stack of panels struck by a sound wave at oblique incidence. Before considering this case, however, the analysis begins with simpler cases, namely one and two panels in a duct at normal incidence.

In the various two-dimensional cases (two panels in a duct, and the infinitely tall stack of panels with either constrained or general panel height), finding expressions for the average power transmitted follows the same steps. First, an assumption is made about the acoustic barrier such as its velocity (U_a and U_b for two panels in a duct) or its displacement (the amplitudes A and B of the heaviside functions for the infinite stack of panels). Expressions for these terms are eventually determined, but the calculation begins assuming they are known. Next the wave and momentum equations are used to find expressions for acoustic pressure and velocity. Boundary conditions at $x = 0$ are used to find the acoustic pressure and velocity in terms of the assumed parameter. An expression for the assumed parameter is acquired by relating the force on a panel and its velocity to the mechanical impedance. Power is the product of the real part of the now-complete expressions for acoustic pressure and velocity.

By plotting the average transmitted power vs $\bar{\omega}$, it can be concluded how the panel dynamics should be adjusted to minimize sound transmission. For both the cases of two panels in a duct and an infinite stack of panels, the panels should have low fluid loading, large structural damping, large panel height relative to the wavelength of the sound source, and panels with very different natural frequencies. For two panels in a duct, the cut-on frequency $\omega_{c/o}$ should be less than both ω_R and 1. For the infinite stack of panels, the incident angle θ should be large (more oblique angles are preferred).

Much work still remains before the acoustic barrier in Figure 1 is completely understood. Even at its most generalized point, this project was confined to two-dimensions and every other panel was identical. There may be interesting canceling behaviors when every fourth panel instead is identical, for example, or possibly when every panel is dynamically unique.

Also, in the future, this project could be expanded to include experiments at oblique incidence to test the predictive plots of average power vs $\bar{\omega}$. Real physical sound sources produce acoustic pressure waves at a range of incidence angles, so this project could be extended to compute the transmitted power for a broadband random incidence field. Thus, while this analysis has been entirely theoretical, there are interesting implications that are worth exploring further, both theoretically and physically.

6 Appendix

6.1 Steps between Equation 11 and 12

$$m\ddot{x} = -R\dot{x} - s_c x + F_{\mathrm{I}} - F_{\mathrm{II}} \tag{154}$$

$$\left(-m\omega^2 + i\omega R + s_c\right) x = F_{\mathrm{I}} - F_{\mathrm{II}} \tag{155}$$

$$\left(i\omega m + R - \frac{is_c}{\omega}\right) i\omega x = \left(i\omega m + R - \frac{is_c}{\omega}\right)\mathbf{u} = F_{\mathrm{I}} - F_{\mathrm{II}} \tag{156}$$

$$\mathbf{Z_m} = \frac{F_{\mathrm{I}} - F_{\mathrm{II}}}{\mathbf{u}} = R + i\left(m\omega - \frac{s_c}{\omega}\right) \tag{157}$$

6.2 Calculation of Equations 37 and 38

$$\begin{aligned} \hat{P}_0 = \frac{1}{2H}\int_{-H}^{H} U_s(y)\,\mathrm{d}y = \frac{1}{H}\int_{0}^{H} U_s(y)\,\mathrm{d}y = \frac{1}{H}\left[\int_0^{\frac{H}{2}} U_b\,\mathrm{d}y + \int_{\frac{H}{2}}^{H} U_a\,\mathrm{d}y\right] \\ = \frac{U_a + U_b}{2} \end{aligned} \tag{158}$$

$$\begin{aligned} \hat{P}_{m>0} &= \frac{1}{H}\int_{-H}^{H} U_s(y)\cos\left(\frac{m\pi y}{H}\right)\mathrm{d}y \\ &= \frac{2}{H}\int_{0}^{H} U_s(y)\cos\left(\frac{m\pi y}{H}\right)\mathrm{d}y \\ &= \frac{2}{H}\left[\int_0^{\frac{H}{2}} U_b\cos\left(\frac{m\pi y}{H}\right)\mathrm{d}y + \int_{\frac{H}{2}}^{H} U_a\cos\left(\frac{m\pi y}{H}\right)\mathrm{d}y\right] \\ &= \frac{2}{H}\left[\frac{HU_b}{m\pi}\sin\left(\frac{m\pi y}{H}\right)\Big|_0^{\frac{H}{2}} + \frac{HU_a}{m\pi}\sin\left(\frac{m\pi y}{H}\right)\Big|_{\frac{H}{2}}^{H}\right] \\ &= \begin{cases} 0 & \text{even m} \\ \frac{2}{m\pi}\sin\left(\frac{m\pi}{2}\right)(U_b - U_a) & \text{odd m} \end{cases} \\ &= \frac{2}{(2m-1)\pi}(-1)^{m+1}(U_b - U_a) \end{aligned} \tag{159}$$

6.3 Derivation of Equation 52 from 40

The m=1 mode cuts on when k_{x_m} first becomes imaginary or when $k = k_{\mathrm{c/o}} = \frac{\pi}{H}$. $\omega = ck$ so $\omega_{\mathrm{c/o}} = \frac{\pi c}{H}$

$$\begin{aligned} \bar{k}_{x_m} = Hk_{x_m} = -iH\sqrt{\left(\frac{(2m-1)\pi}{H}\right)^2 - k^2} = -\frac{iH}{c}\sqrt{(2m-1)^2\omega_{\mathrm{c/o}}^2 - k^2} \\ = -\frac{i\pi}{\omega_{\mathrm{c/o}}}\sqrt{(2m-1)^2\omega_{\mathrm{c/o}}^2 - k^2} = -i\pi\left(\bar{\omega}_{c/o}\right)^{-1}\sqrt{(2m-1)^2\bar{\omega}_{c/o}^2 - \bar{\omega}^2} \end{aligned} \tag{160}$$

7 Acknowledgements

I want to thank Dr. Donald Bliss for his guidance and encouragement, and Duke Mechanical Engineering PhD candidate Mauricio Villa for his insight and constant selflessness.

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[1] "Reduction of Sound Transmission through Fuselage Walls by Alternate Resonance Tuning (ART)", D. B. Bliss and J. A. Gottwald, presented at the AIAA 12th Aeroacoustics Conference, San Antonio, TX, (April 10-12, 1989).

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