Magnetic Field Evolution in Magnetars with Gravitational Wave Applications

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Abstract

With the recent gravitational wave detection by LIGO, many additional sources of gravitational wave signals are being explored. One of these is highly magnetic neutron stars known as magnetars. We use the numerical code PLUTO to simulate magnetic fields in magnetars and their development over Alfven time scales in order to determine whether rotating magnetars have potential to produce detectable signals by LIGO. The form of a stable magnetic field configuration in magnetars is not yet known, but is actively being researched, as it is applicable to many astrophysical problems (not just limited to gravitational wave source modeling). We numerically model a magnetar as an \( n=1 \) polytrope using a 50x20x20 spherical grid, time step \( t = 10^{-8} \text{s} \) and a maximum field strength of \( 5 \times 10^{15} \text{ G} \). Purely poloidal and purely toroidal field configurations and their evolutions are investigated, finding that the wave strain produced is sufficient to make certain neutron stars candidates for LIGO.

Boundary conditions for these stars, and their implementation within the code PLUTO, are discussed, as well as the effects of resolution and field strength on our simulation. We comment on the definition of our poloidal field with respect to stability, and investigate the effects of perturbations and instabilities on field evolution. We see that an \( m=2 \) instability along the neutral line of a poloidal field will induce a twisted torus configuration that remains fairly consistent over hundreds of Alfven times. This is one of the first simulations at these field strengths that appears to remain stable over several hundred timescales, indicating that the final
field configuration could be that of a magnetar in nature.
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1 Introduction

1.1 Gravitational Waves

Albert Einstein first postulated gravitational waves in 1915 as a consequence of his theory of gravity. Gravitational waves, as we understand them, arise when a massive body undergoes an acceleration, provided the motion is not spherically or cylindrically symmetric, and are the result of a time varying quadrupole. These waves are in the form of three types of signals:

1. Inspirals, which describe a signal increasing in frequency as two objects coalesce, such as from a system of binary black holes or neutron stars

2. Bursts, which characterize a short non-repeating signal, and are the product of such things as supernovae explosions

3. Continuous signals, which are steady and consistent and from such sources as rotating asymmetric stars.

Though gravitational waves are no challenge to produce, detection is far from trivial. For one, their wavelengths are on the order of kilometers long, requiring interferometers that are also kilometers long for detection. Additionally, the amplitudes of the strongest signals we expect to observe on Earth will have a wave strain ($h$), a
parameter which describes the amplitude, at $h = 10^{-20}$ or less. This is enough to distort the shape of the Earth by $10^{-13}$ meters, or about 1% of the size of an atom. By contrast, the (nonradiative) tidal field of the Moon raises a tidal bulge of about 1 meter on the Earth’s oceans[1]. Clearly, these signals are so weak that it would take an extraordinarily dramatic event, such as the collision of two black holes, to produce something detectable with our current apparatuses.

Gravity has many interesting properties, reaching far beyond what was originally formulated by Newton. According to Einstein’s general theory of relativity, gravity is how mass deforms the shape of spacetime. This deformation can propagate throughout the Universe, alternately stretching and compressing space as it goes [2].

The detection of these waves is important for science in several respects. For one, it confirms Einstein’s theory of gravity, which, while universally accepted as the formalism that governs large massive bodies, lacked this critical experimental verification. More importantly, being able to detect gravitational waves opens up a new medium through which we can image the universe. This is particularly interesting because gravitational waves can pass through barriers unaltered, whereas light, our main method of viewing the universe, has the potential to be scattered. This will allow us to see parts of the Universe that were previously invisible, such as the interiors of stars.

To show how gravitational plane waves arise mathematically, we follow the approaches of introductory texts by James Hartle and Sean Carrol[3][4]. We first look to the weak-field approximation, in which we decompose our metric into the flat minkowski metric and a small perturbation.

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}
\]  

(1.1)

We assume $|h_{\mu\nu}| \ll 1$, and can thus ignore terms $O(h^2)$. From here we obtain the
equation

\[ g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \]  (1.2)

We must compute connection coefficients, for they will be necessary to calculate our tensors:

\[ \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} (g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}) = \frac{1}{2} \eta^{\rho\lambda} (h_{\mu\lambda,\nu} + h_{\nu\lambda,\mu} - h_{\mu\nu,\lambda}) \]  (1.3)

Computing our Riemann, Ricci and thus Einstein Tensors, we obtain:

Riemann Tensor:

\[ R_{\mu\nu\rho\sigma} = \frac{1}{2} \left( \partial_\rho \partial_\nu h_{\mu\sigma} + \partial_\sigma \partial_\mu h_{\nu\rho} - \partial_\sigma \partial_\nu h_{\mu\rho} - \partial_\rho \partial_\mu h_{\nu\sigma} \right) \]  (1.4)

Ricci Tensor:

\[ R_{\mu\nu} = \frac{1}{2} \left( \partial_\sigma \partial_\nu h^\sigma_{\mu} + \partial_\sigma \partial_\mu h^\sigma_{\nu} - \partial_\nu \partial_\mu h - \Box h_{\mu\nu} \right) \]  (1.5)

Where \( \Box = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2 \)

Einstein Tensor:

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R = 8\pi G T_{\mu\nu} \]  (1.6)

In our weak field approximation, \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \) is not uniquely defined—our perturbation can be different in other coordinate systems. To address this and eliminate degeneracy, we must fix a gauge. In this case we will be choosing the transverse gauge, which is analogous to the Coulomb gauge in electromagnetism.

We write \( h_{\mu\nu} \) as a decomposition of its trace and trace-free parts:

\[ h_{00} = -2\Phi \]  (1.7)

\[ h_{0i} = \omega_i \]  (1.8)

\[ h_{ij} = 2s_{ij} - 2\psi \delta_{ij} \]  (1.9)
In this gauge, our Einstein tensor becomes:

\[ G_{00} = 2\nabla^2 \psi = 8\pi G T_{00} \quad (1.10) \]

In weak field limit, we take \( T_{\mu\nu} = 0 \). We obtain the equations \( \nabla^2 \psi = 0, \nabla^2 \omega_j = 0, \nabla^2 \phi = 0, \mathbf{\Box} s_{ij} = 0 \)

We now have our transverse traceless gauge:

\[
h^{TT}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2s_{ij} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

We use \( h^{TT}_{\mu\nu} \) for its convenience when comparing with other resources. It is purely spacial, traceless and transverse. Our equation of motion is then

\[ \Box h^{TT}_{\mu\nu} = 0 \quad (1.11) \]

and it follows that

\[ \partial_\mu h^{TT}_{\mu\nu} = 0. \quad (1.12) \]

It is from this equation that our plane wave solutions arise. Looking at equation 12, we see that a solution is

\[ h^{TT}_{\mu\nu} = C_{\mu\nu} e^{ik_\sigma x^\sigma} \quad (1.13) \]

where \( C_{\mu\nu} \) is a constant, symmetric and purely spacial tensor, and \( k^\sigma = (\omega, k^1, k^2, k^3) \) is the wave vector. By plugging our solution back into the equation of motion, we obtain the condition \( k_\sigma k^\sigma = 0 \), which tell us that the plane wave is a solution if the wave vector is null. To ensure that our perturbation is transverse, we require

\[ 0 = \partial_\mu h^{TT}_{\mu\nu} = iC^{\mu\nu} k_\mu e^{ik_\sigma x^\sigma} \quad (1.14) \]
so,

\[ 0 = C^{\mu \nu} k_\mu. \] (1.15)

To simplify our solution, let’s consider a wave traveling in the \( x^3 \) direction:

\[ k^\sigma = (\omega, 0, 0, k^3) = (\omega, 0, 0, \omega). \] (1.16)

We can determine that \( k^3 = \omega \) because our wave vector is null. We can from there derive that \( C_{3\nu} = 0 \) and thus that the only non-zero components of our symmetric, traceless \( C_{\mu \nu} \) are \( C_{11}, -C_{11}, C_{12} \) and \( C_{21} \). Thus, we see in general that

\[
C_{\mu \nu} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & C_{11} & C_{21} & 0 \\
0 & C_{12} & -C_{11} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

This tells us that for a plane wave in this transverse gauge, only two components (outside of the frequency), \( C_{11} \) and \( C_{12} \) characterize the plane wave behavior. Having covered the mathematical origins of gravitational plane waves, the next section discusses the research being done to detect them.

1.2 LIGO

The Laser Interferometer Gravitational-Wave Observatory (LIGO) is the leading collaboration in the gravitational wave hunt. The LIGO experiment uses a pair of ground based interferometers in the United States, one in Livingston, Louisiana and the other in Hanford, Washington to isolate correlated signals, some of which they hope to identify as originating from gravitational waves. The mirrors are 11kg, the arms 4km long and the laser is powered at 10 Watts. LIGO can measure frequencies between 10Hz and 10,000Hz and a wave strain (h) of roughly \( 10^{-23} \). The wave strain sensitivity thresholds are shown in figure 1.1.
With the introduction of Advanced LIGO in Fall of 2015, the laser power increased to 200 W, the test mass objects are almost one third larger in diameter, and the frequency cutoff has moved from 40Hz down to 10Hz. The observable volume of space of advanced LIGO is 1000 times greater than that of Initial LIGO, allowing better access to potential sources. “LIGO is designed to detect a change in distance between its mirrors 1/10,000th the width of a proton! This is equivalent to measuring the distance to the nearest star to an accuracy smaller than the width of a human hair!” [6].

On September 14, 2015, Advanced LIGO saw the merging of two inspiraling black holes—an event which occurred 1.3 billion years ago[7]! This was the first event of its kind to be detected, and ushers in a new era of gravitational wave astronomy. The effort continues to find new signals from a variety of different sources.
2.1 Magnetic Field Stability in Ap/Bp Stars

Ap, Bp stars are peculiar A and B stars which have an abundance of certain metals and a much slower rotation than typical A, B stars. They have large magnetic fields ranging from 0.03-3 Tesla. Several models of these stars have been created, though they fail to describe many of our observations accurately, which suggests that they have a complex field structure. Much of the work on stable fields in these stars comes from Braithwaite [8][9][10][11], who uses numerical simulations to look at stability of Ap stars on Alfven time scales, or the time scale on which a magnetic wave crosses a star.

To examine field configurations and features in these types of stars, the procedure followed is generally the same: begin with some reasonable initial state based on what is known from observation and time evolve it numerically. If the magnetic field settles into a stable configuration after several Alfven time scales, our system is considered stable. By this method, the same stable field configuration is always found: a nearly asymmetrical torus inside the star with toroidal and poloidal com-
Stable magnetic fields have both toroidal and poloidal components (shown in figure 2.1). The torus can be either right or left handed depending on initial conditions. These general configurations are not affected by initial conditions, but the surface field strength does depend on them.

Much of the work done on fields in Ap and Bp stars can be applied to highly magnetized neutron stars.

2.2 Neutron Stars and Magnetars

Neutron stars are some of the most dramatic phenomena in our universe. With a radius on the order of 10km, they can be twice as massive as our sun. Magnetars are highly magnetic neutron stars, with fields that can reach above $10^{15}$ Gauss at the surface and are thought to be even stronger in the interior. This is particularly impressive when one considers that the magnetic field of the Earth is under 1 Gauss. These strong magnetic fields deform the star as a result of the Lorentz forces they exert

$$\vec{F}_{\text{Lorentz}} = q\vec{v} \times \vec{B}. \tag{2.1}$$
Figure 2.2: The evolution of a magnetic field in a stable Ap star at times $t = 0$ days (top row figures) 0.18 days (middle left figure), 0.54 days (middle right figure) and 5.4 days (bottom row figures, with the right figure showing the star along the axis), as calculated by Braithwaite and Nordlund [8].
If the rotation axis and magnetic axis of a magnetar are offset, gravitational waves can be produced.

The magnetic fields in these stars were probably already present at birth. Their origins are not yet known, and determining them is an active area of research in astrophysics. Energy is released by rearrangement of the magnetic field configuration in the star and as the rotation of the star slows over time. They release energy over a timescale of around $10^4$ years, which is much greater than the Alfven time scale over which unstable fields evolve (0.1s). They have ample time to either evolve into a stable configuration or decay to nothing before being frozen in by the crust. In general, it has been found that an arbitrary unstable initial field does not decay completely, but gets stuck in a stable equilibrium at some magnitude [8].

For gravitational wave detection from neutron star sources it is important to understand the star’s magnetic field configuration. If the magnetic and rotational axes are not aligned (figure 2.3), the deformation due to the magnetic field will result in asymmetrical rotation. This will create a time varying quadrupole and, as the star spins down over time, consequently gravitational radiation.

The wave strain ($h$) resulting from such a configuration is fairly straightforward
to calculate. We first calculate the moments of inertia \( I \). From there we can obtain

a quantity known as ellipticity \( \epsilon \), which measures how much a star is stretched or

squeezed, and is directly proportional to \( h \) [12][13].

\[
\epsilon = \frac{I_{zz} - I_{xx}}{I_0} \quad (2.2)
\]

\[
I_{jk} = \int_V \rho(r)(r^2 \delta_{jk} - x_j x_k) dV \quad (2.3)
\]

\[
h = \frac{16\pi^2 G \epsilon I_0 f^2}{c^4 r} \quad (2.4)
\]

Here, \( I_{xx} \) and \( I_{zz} \) are the moments of inertia with respect to the rotational and

magnetic axes respectively.

Though scientists have observed 2,000 neutron stars in our Milky Way, our targets

for gravitational wave emission are newly born stars. This is because they will have

periods that correspond to detectable frequencies, unlike the magnetars we know of

which have frequencies too low for LIGO to detect (below 10Hz). As these young

stars emit gravitational radiation, they lose energy causing them to spin down to

frequencies below this threshold.

2.3 Alfven Time

The Alfven crossing time is the time it takes for a magnetic wave to cross a star.

This is the time scale that we are interested in for our simulations, as it is the time

scale on which either the star will stabilize or our simulation will break down and no

longer be able to run, indicating an unphysical situation (such as the emergence of

negative density). It is straightforward to calculate the Alfven speed:

\[
v_A = \frac{B_0}{\sqrt{\mu_0 \rho_0}} \quad (2.5)
\]
and from there the Alfven crossing time:

\[ t_A = \frac{d}{v_A} \tag{2.6} \]

Noticing that \( v_A \propto \frac{1}{\sqrt{\rho_0}} \), we see that waves will take infinite time to propagate in a vacuum. Thus, we can anticipate problems calculating with \( \rho = 0 \). It is critical to consider this when modeling a neutron star, as our code will have difficulty processing a perfect vacuum beyond the boundary of the star. Thus, we set \( \rho = 10^{10} g/cm^3 \) outside, as this is the smallest value for which it doesn’t get numerically truncated to 0. This exterior density is several orders of magnitude weaker than the density within the star, rendering it effectively a vacuum.

It is important to understand the Alfven time scale of our system in order to know when we should look for stability, or alternatively, when we should expect instabilities to arise. We can directly calculate this using equations 2.5 and 2.6. We use the software VisIT to calculate and plot the results.

2.4 Magnetohydrodynamic Equations and Flux Freezing

To consider the problem of gravitational emission from a magnetic neutron star, it is important to be acquainted with the magnetohydrodynamic equations. The ideal MHD equations, which are composed of the HD equations and Maxwell’s equations, are as follows [14]:
MHD Equations:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \quad (2.7a)
\]
\[
\frac{d}{dt} \left( \frac{P}{\rho^\gamma} \right) = 0 \quad (2.7b)
\]
\[
\rho \frac{d\vec{v}}{dt} + \nabla P + \frac{1}{\mu_0} \left( \nabla \times \vec{B} \right) \times \vec{B} = 0 \quad (2.7c)
\]
\[
\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{v} \times \vec{B}) = 0 \quad (2.7d)
\]
\[
\nabla \cdot \vec{B} = 0 \quad (2.7e)
\]

where \( \rho \) is density, \( \vec{v} \) is fluid velocity, \( P \) is pressure, \( \vec{B} \) is magnetic field, and \( \gamma \) is the ratio of specific heats \( C_p/C_v \).

These are a set of five partial differential equations describing the physical properties of plasma. These are particularly applicable to astrophysics and cosmology, as 99% of the baryonic matter content of the universe is made up of plasma.

From these equations, we can see mathematically how permanent magnetic field lines can become frozen into the crust of the star through a process known as “flux freezing”. This concept is important to our discussion as it explains why magnetars can retain their high field strengths.

We begin with MHD Ohm’s Law

\[
\vec{E} + \vec{v} \times \vec{B} = 0
\]  

(2.8)

Next we recall Faraday’s Law:

\[
\Phi = \int_{s} \vec{B} \cdot ds
\]

(2.9)

Differentiating Faraday’s Law in two parts and applying Stokes Theorem, we obtain:
\[
\frac{\partial \Phi}{\partial t} = (\frac{\partial \Phi}{\partial t})_1 + (\frac{\partial \Phi}{\partial t})_2
\]  
(2.10a)

where,
\[
(\frac{\partial \Phi}{\partial t})_1 = \int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} = -\int_s \nabla \times \vec{E} \cdot d\vec{s}
\]  
(2.10b)

\[
(\frac{\partial \Phi}{\partial t})_2 = \int \vec{B} \cdot \vec{v} \times d\vec{l} = \int \vec{B} \times \vec{v} \cdot d\vec{l} = \int \nabla \times (\vec{B} \times \vec{v}) \times d\vec{s}
\]  
(2.10c)

\[
(\frac{\partial \Phi}{\partial t})_2 = \int \vec{B} \cdot \vec{v} \times d\vec{l} = \int \vec{B} \times \vec{v} \cdot d\vec{l} = \int \nabla \times (\vec{B} \times \vec{v}) \times d\vec{s}
\]  
(2.10d)

Combining our two equations back together, we find
\[
(\frac{\partial \Phi}{\partial t}) = -\int_s \vec{E} + \vec{v} \times \vec{B} \cdot d\vec{s}.
\]  
(2.11)

Recalling that
\[
\vec{E} + \vec{v} \times \vec{B} = 0
\]  
(2.12)

we see,
\[
\frac{\partial \Phi}{\partial t} = 0.
\]  
(2.13)

From this, we can determine that the flux remains constant at every contour, and thus the field lines move with the plasma—they are “frozen in”!
3.1 PLUTO

PLUTO is a C based grid code that solves the MHD equations. “[It is] is a freely-distributed software for the numerical solution of mixed hyperbolic/parabolic systems of partial differential equations (conservation laws) targeting high Mach number flows in astrophysical fluid dynamics. The code is designed with a modular and flexible structure whereby different numerical algorithms can be separately combined to solve systems of conservation laws using the finite volume or finite difference approach based on Godunov-type schemes. Equations are discretized and solved on a structured mesh that can be either static or adaptive”[15]. We use PLUTO for all of our simulations.

A major drawback to PLUTO is that it does not solve the Poisson equation. Rather, the user is required to manually enter in appropriate gravitational potentials. For instructions on how to set up PLUTO on a Mac computer, see Appendix 1.

For my work, PLUTO was used to model a stable neutron star as an n=1 polytrope, first in hydrostatic equilibrium and then with addition of magnetic fields. The code is used in conjunction with the program VisIT, which turns the numerical sim-
ulations into visualizations. We initially used a grid with 20 grid points in the radial, \( \theta \) and \( \phi \) directions for our simulations, to provide enough resolution to extract useful information, but still limit computing time. Later, this resolution was updated to include a finer grid, as discussed in Chapter 4.

### 3.2 VisIT

We use the application VisIt to visualize and analyze our data. VisIt is an Open Source, interactive, scalable, visualization, animation and analysis tool from Lawrence Livermore National Laboratory. It was originally developed by the Department of Energy Advanced Simulation and Computing Initiative to examine results from terascale simulations. VisIT reads .vtk files, which PLUTO can be made to output with edits to the ‘pluto.ini’ file.

### 3.3 Polytropes

Many stars are well modeled as polytropes, and thus are described by the polytropic equation of state:

\[
P = K \rho^{1+1/n}.
\]  

(3.1)

To determine the initial conditions that will result in a stable star, we begin by considering a star in hydrostatic equilibrium, balancing \( F_{\text{Gravity}} \) and \( F_{\text{Pressure}} \).

\[
\frac{dP(r)}{dr} = -\rho(r) \frac{Gm(r)}{r^2}
\]

(3.2)

with

\[
\frac{dm(r)}{dr} = 4\pi r^2 \rho(r)
\]

(3.3)
as the mass contained within radius r. Taking the divergence of equation 3.2 and plugging in equation 3.3, we obtain:

\[
\frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2}{\rho} \frac{d\rho}{dr} \right) = -4\pi G \rho
\]  

(3.4)

We insert our polytropic equation of state, and with the following change of variables:

\[
r = \xi \alpha \\
\rho = \Theta^n \rho_c
\]  

(3.5)

where,

\[
\alpha^2 = K \frac{n+1}{4\pi G} \rho_c^{(1-n)/n}.
\]

This gives us a greatly simplified expression, known as the Lane-Emden Equation:

\[
\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\Theta}{d\xi} \right) = -\Theta^n
\]  

(3.6)

with initial conditions \( \Theta(0) = 1 \) and \( \Theta'(0) = 0 \).

For the general case of an unspecified n, the Lane-Emden equation cannot be solved analytically and, rather, needs to be tackled numerically. For these instances, a numerical solver has been written in C for the equation, which employs the fourth-order Runge-kutta method. To use this, we break our second order differential equation into the following system of two first order equations:

\[
\frac{d\Theta}{d\xi} = -\frac{z}{x^2}  \\
\frac{dz}{d\xi} = \Theta^n \xi^2.
\]  

(3.7a, 3.7b)

We can rederive \( \rho \) and \( P \) from the results. The full code can be found in Appendix 2.
3.3.1 \( N=1 \) Polytrope

Neutron stars are best modeled as \( n=1 \) polytropes. Fortunately, the \( n=1 \) case of the Lane-Emden equation be solved analytically. We take

\[ P = K\rho^2 \]  

(3.8)
as our equation of state. Then, the Lane-Emden equation becomes

\[ \frac{1}{\xi^2} \frac{d}{d\xi}(\xi^2 \frac{d\Theta}{d\xi}) = -\Theta \]  

(3.9)

We obtain as a solution

\[ \Theta(\xi) = \frac{\sin(\xi)}{\xi} \]  

(3.10)

and returning to our original variables, find our initial conditions in \( \rho \) and \( P \) to be:

\[ \rho(r) = \rho_c \frac{\sin(\pi r/R) R}{r\pi} \quad r < R \]  

(3.11a)

\[ P(r) = K\rho(r)^2 \]  

(3.11b)

For a star of radius \( R=10\text{km} \) and mass \( M=1.4M_\odot \), we obtain constants:

\[ \rho_c = 2.2 \times 10^{15} \text{g/cm}^3 \]  

(3.12)

\[ K = 4.25 \times 10^4 \text{g}^{-1}\text{cm}^5\text{s}^{-2} \]  

(3.13)

Giving PLUTO solely these initial conditions will, as expected, result in a star that evaporates, as there is no gravitational potential to hold it together. This is seen in figure 3.1, which shows the density profile of an evaporating polytrope.

This is a positive first result, for it allows us to see that the code is behaving how we expect.

The next step is to introduce an appropriate gravitational potential.
3.3.2 Gravitational Potential

To prevent the evaporation of our star, we need to introduce a gravitational potential. Giving PLUTO a large potential at random, in order to test the code and ensure it is working appropriately, we as expected find a collapsing star, as shown in figure 3.2.

To find the appropriate potential for our $n=1$ polytrope, we begin with the Poisson equation.

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \phi \right) = 4\pi G \rho \quad (3.14)
\]

PLUTO does not have the machinery to solve the Poisson equation, and thus it is our job to derive the gravitational potential and manually type it into the code. To do so, we recall our equation for density $\rho = \frac{\rho_c \sin(\pi r/R)}{r\pi}$, which we use in the Poisson equation to solve for the potential.

\[
\phi = 4G\rho_c R \int \frac{dr}{r^2} \left[ \int r \sin(\pi r/R) dr \right] = 4G\rho_c \left( \frac{-R^3 \sin(\pi r/R)}{\pi^2 r} + \frac{C_1}{r} + C_2 \right) \quad (3.15)
\]

with

\[
\phi(0) = 0 \quad \phi(R) = -\frac{GM}{R} \quad (3.16)
\]
To avoid a singularity at $R=0$, we need to take $C_1=0$. Matching conditions at the boundary, we obtain $C_2=-\frac{M}{4R\rho_c}$. After solving for constants, we still have $\frac{1}{r}$ behavior at the center, which can be remedied by taking the limit of $\frac{\sin(\pi r/R)}{r}$ as $r$ approaches 0, which is $\frac{\pi}{R}$, and replacing it in our expression for $\phi_0$. We now have our potential:

\begin{align*}
\phi_0 &= 4G\rho_c\left(-\frac{R^2}{\pi}\right) - \frac{M}{4R\rho_c} \\
\phi_{\text{inside}} &= 4G\rho_c\left(-\frac{R^3}{\pi^2r}\right) - \frac{M}{4R\rho_c} \\
\phi_{\text{outside}} &= -\frac{GM}{r}
\end{align*}

(3.17a) (3.17b) (3.17c)
3.4 Hydrostatic Equilibrium

With these initial conditions, we are able to model a neutron star (as an n=1 polytrope) in hydrostatic equilibrium.

Initial Conditions:

\[
\rho(r) = \rho_c \frac{\sin(\pi r/R)}{r \pi} \quad r < R \tag{3.18a}
\]

\[
P(r) = K \rho(r)^2 \tag{3.18b}
\]

\[
\phi_0 = 4G\rho_c(-\frac{R^2}{\pi}) - \frac{M}{4R\rho_c} \tag{3.18c}
\]

\[
\phi_{\text{inside}} = 4G\rho_c(\frac{-R^3 \sin(\pi r/R)}{\pi^2 r} - \frac{M}{4R\rho_c}) \tag{3.18d}
\]

\[
\phi_{\text{outside}} = -\frac{GM}{r} \tag{3.18e}
\]

![Figure 3.3: The density and velocity profiles, respectively, for a star in hydrostatic equilibrium](image)

Through time scales of 1s in PLUTO, we see no changes beyond small oscillations in both the density and velocity profiles. The velocity profile makes sense physically: we expect some very slight movement in the interior that is essentially constant throughout. Once our star is stable, we introduce a magnetic field.
3.5 Magnetic Field

As mentioned previously, for gravitational wave detection it is critical to understand the magnetic field configurations of stable stars. This will allow us to determine how they deform and, if rotating, what wave strain we can expect as a consequence. Though rotation is critical for the star to emit gravitational radiation, we will neglect its effects as they pertain to the magnetic field, in order to simplify our model and calculations. This is a reasonable approximation, as magnetars rotate slowly.

To construct a magnetic field, we closely follow the work of Haskell, Samuelsson, Glampedakis and Andersson (2008). We begin with the equation for magnetohydrostatic equilibrium.

\[
\nabla \rho + \nabla \phi = \frac{(\nabla \times \vec{B}) \times \vec{B}}{4\pi \rho} = \frac{\vec{L}}{4\pi \rho} \tag{3.19}
\]

Where \(\vec{L}\) is the Lorentz force, and the gravitational potential \(\phi\) obeys the Poisson equation. From Maxwell’s equations, we know that,

\[
\nabla \cdot \vec{B} = 0 \tag{3.20}
\]

and that we are considering a barytropic equation of state, meaning \(\rho = \rho(P)\).

Taking the curl of equation 3.19, we obtain

\[
\nabla \times \left[ \frac{\vec{B} \times (\nabla \times \vec{B})}{\rho} \right] = 0 \tag{3.21}
\]

which is a constraint our field must satisfy. Whichever field is arrived upon for the interior of the star must also match the exterior solution for a field:

\[
\vec{B}_{\text{ext}} = \sum_{l\geq m} -(l + 1) \frac{A_l}{r^{l+2}} Y_l^m \hat{r} + \frac{A_l}{r^{l+2}} \hat{\phi} Y_l^m \hat{\phi} + \frac{im A_l}{r^{l+2}} \sin(\theta) Y_l^m \hat{\theta} \tag{3.22}
\]

We thus have constraints on our initial field configurations. We begin by exploring a purely poloidal field, using the dimensionless radius \(y = \frac{\pi r}{R}\). Inside of the star, the
magnetic field is described by:

\[
\vec{B}_r = \frac{B_s \cos(\theta)}{\pi (\pi^2 - 6)} [y^3 + 3(y^2 - 2)\sin(y) + 6\cos(y)]
\] (3.23a)

\[
\vec{B}_\theta = \frac{B_s \sin(\theta)}{2\pi (\pi^2 - 6)} [-2y^3 + 3(y^2 - 2)(\sin(y) - \cos(y))]
\] (3.23b)

\[
\vec{B}_\phi = 0
\] (3.23c)

where \(B_s\) is our maximum field strength. Taking \(l = 1\) and \(m = 0\), we obtain our conditions outside of the star:

\[
\vec{B}_r = \frac{B_s R^3 \cos(\theta)}{r^3}
\] (3.24a)

\[
\vec{B}_\theta = \frac{B_s R^3 \sin(\theta)}{2r^3}
\] (3.24b)

\[
\vec{B}_\phi = 0.
\] (3.24c)

Though we began using a maximum field strength of \(10^{17}\)G, it soon became apparent that this is too strong to be physical, because it leads to a Beta parameter of nearly 1, as discussed in more detail in the next section. Instead, we opt for a maximum field strength of \(5 \times 10^{16}\)G, which is still on the high end of what we would expect, but in terms of magnetic pressure much more reasonable. The large field strength reduces our Alfven time and allows for simulations to complete more rapidly.

We will also explore the evolution of an initially toroidal magnetic field:

Inside star:

\[
\vec{B}_r = 0
\] (3.25a)

\[
\vec{B}_\theta = 0
\] (3.25b)

\[
\vec{B}_\phi = \frac{B_s \sin\sin\theta}{\pi}
\] (3.25c)

\[
(3.25d)
\]
Outside star:

\[
\begin{align*}
\vec{B}_r &= 0 \\
\vec{B}_\theta &= 0 \\
\vec{B}_\phi &= 0.
\end{align*}
\]

(3.25e) (3.25f) (3.25g)

We again take a maximum field strength of \( B_s = 5 \times 10^{16} \text{G} \).

When simulating in PLUTO, we must make the unphysical approximation of giving the vacuum outside of the star a density. This is because, as we know from our previous discussion of time scales, when we have a magnetic field in a vacuum we will be dividing by zero for our time. Thus, without a density of \( 10^{10} \text{g/cm}^3 \) on the exterior of the star, our code will stop running almost immediately. While this approximation does prevent our model from being entirely realistic, it is fair, as the interior of the star is still more dense by five orders of magnitude than the exterior.

3.5.1 Preliminary Field Evolution

Simulations of poloidal and toroidal fields with these expressions and PLUTO’s preset boundary conditions can be seen in figures 3.4 and 3.5. In both instances, the field configuration hardly changes in time. We see an increase in maximum field strength in the poloidal case, as well as some deformation in the star (illustrated by the changing density profile), but no visible change in the actual structure of the field. There is even less change in the poloidal case. The limited field evolution is indication that there is an issue with the code’s implementation of the problem, as we expect initially toroidal and poloidal fields to grow additional components with time in order to stabilize. We will address this going forward by exploring alternate boundary conditions and grid resolution, and triggering evolution with velocity perturbations.
Figure 3.4: Preliminary poloidal field evolution. Shown are streamlines to illustrate field configurations and the density profile to show slight deformation. The figures on the left correspond to $t = 0s$ and on the right correspond to $t=.1s$. 
Figure 3.5: Preliminary toroidal field evolution. Shown are streamlines to illustrate field configurations and the density profile to show how static the mass distribution remains. The figures on the left correspond to t = 0s and on the right correspond to t = .1s.
When setting up simulations, it is critical to understand the behavior of different parameters at all of the boundaries. PLUTO comes with a set of predefined boundary conditions, which are not applicable to every simulation. It was through lengthy investigation and plenty of trial and error that we arrived at boundary conditions that we feel well suit our specific problem.

4.1 Grid & Resolution

We define our star with a spherical coordinate system. This may seem like a natural choice, given the geometry of the problem, but in fact the bulk of the literature that tackles these problems computationally uses a cartesian grid to avoid degeneracy at $\phi = 0, 2\pi$. PLUTO offers the option of a spherical mesh, which we take to simplify the math.

Our simulations of a poloidal field were initially run with 20 grid points in $r, \theta$ and $\phi$, and implemented ‘axissymmetric’ boundary conditions in $\theta$. The definition of these conditions as given in the user manual is in figure 4.1. It soon became apparent that this boundary condition was inappropriate, for it created a highly unphysical
density profile within the star—the star’s mass became concentrated in triangular regions to allow for a region with the density of free space to emerge along the Z-axis (figure 4.2).

Using the same boundary conditions, an identical simulation was run with a resolution increase in the radial direction from 20 to 40 grid points. The results produced are not of great physical consequence, but are informative from the perspective of learning about how PLUTO handles different grids. Both simulations stopped running on the order of $t = 10^{-5}s$, due to the development of negative density—not surprising in a star with improper boundary conditions. There was, however, a noticeable difference in run time and field development between the two. The lower resolution simulation stopped running at $t = 1.3 \times 10^{-5}s$, while the higher resolution
Figure 4.3: The effects of resolution on simulations. Both show the magnetic field magnitude in a poloidal simulation with axisymmetric boundary conditions implemented in $\theta$ at time $t = 1.3 \times 10^{-6}$. The left has 50 grid points in $r$ and 20 in $\theta$ and $\phi$, and the right has 20 grid points in $r$, $\theta$ and $\phi$. Both plots use an identical log scale ranging from $5 \times 10^{15} g/cm^3$ to $10^{18} g/cm^3$.

Simulation stopped at $t = 1.56 \times 10^{-5}s$. At $1.3 \times 10^{-5}s$, the difference in field distribution is shown in figure 4.3, with the average field in the lower resolution simulation over 1.5 times greater in magnitude than the average field in the higher resolution simulation ($3.41685 \times 10^{17} G$ compared with $2.18039 \times 10^{17} G$). It is not surprising that they are different in magnitude, as the higher resolution grid allows subtleties in the field to be better expressed.

The effect of resolution on simulations is important to note. While a $20 \times 20 \times 20$ grid was initially used to shorten simulation run time, it is clear that higher resolution is important, and thus we will adopt a $50 \times 20 \times 20$ grid for many of our future simulations, which should be sufficient for expressing relevant field dynamics. Future work could include further analysis of the affect of resolution, to perfect the balance between an efficient run and perfectly accurate results. With the additional computational cost associated with higher resolution simulations, it became apparent that we needed to use a more efficient machine than my Macbook laptop for this investigation.
4.2 Multicore & MASSIVE

To decrease simulation run time, accounts were set up to allow remote access to Desktops at Duke University and at the University of Melbourne as well as at the computing cluster MASSIVE in Australia. In order to work most effectively, I customized a makefile within PLUTO that allowed simulations to run in parallel using multiple CPUs. The makefiles for both the Melbourne machines and MASSIVE were only slightly different, and the code used for the desktops is shown in figure 4.4. The file containing this code needs to be added into the PLUTO/Config directory.

4.3 Boundary Conditions

After a laborious investigation and a period of trial and error with PLUTO’s pre-set boundary conditions for the $\theta$ beginning and end boundaries, we resolved to define them independently. We determined that at the $\theta$ boundary, which lies along the central axis of the star, there should be no $\theta$ component in either the magnetic field.
or the velocity. At the radial boundary, the initial field configuration outside of the star was used as a constraint.

The implementation of these new conditions drastically improved the quality of the simulations, and we can now do reliable physics. Our boundary condition investigations are not over, however. With these new conditions, we find that a poloidal field stabilizes without developing a toroidal component, which we know to be incorrect from Braithwaite’s work. This could be the result of over-constraining the field at the boundary, which forces the field lines back into a poloidal shape at every iteration. To test this, we could omit certain constraints when writing the boundary conditions. The lack of development of a toroidal component could also be a function of the grid resolution—if there are too few grid points, toroidal developments could be entirely obscured.

In the future, we could try expressing the boundary conditions in an entirely new way. Rather than specifying values at $r$, $\theta$ and $\phi$ boundaries, we could extrapolate the internal values and force the gradients to be fixed to the initial gradients. This is a Neumann boundary where the gradient is fixed to some value, rather than the default 0 of outflow [16].

4.4 Our Equations

With our improved boundary conditions, we begin to see interesting field evolutions. The equations for a poloidal that Haskell [17] calculated to be stable have zero magnetic field at the very center of the star. Time evolving these fields, however, results in a purely poloidal star with a field strongest at the center and weakest at the exterior, as shown in figures 4.5 and 4.6. This evolution suggests that the equations of Haskell [17] should be adjusted.
4.4.1 Modifications

In order to more clearly see the early stages of evolution of these stars, we reduced the field strength from $||\vec{B}|| = 10^{17}$ to $||\vec{B}|| = 5 \times 10^{16}$. With the larger field, the $\beta$ parameter, which is defined as

$$\beta = \frac{Pressure}{B_{Pressure}} = \frac{8\pi P}{B^2} \quad (4.1)$$

in cgs units, is too small, or the magnetic pressure is too close to the gas pressure. This results in huge deformations as shown in figure 4.7. Additionally, the smaller field reduces the Alfven time, and thus the star evolves over longer time scales.

From figure 4.7, it is apparent that the $10^{17}$G field used previously is slightly too strong to be common in these types of stars. Thus, it makes more sense to proceed with this weaker field strength. The relative $\beta$ parameters are shown in figure 4.8.
Figure 4.6: The evolution of a poloidal field defined by equations with $\vec{B} = 0$ at the center, as shown with magnetic field vectors (looking down the y-axis of the star). We see the development of a strong field at the center within a few Alfven timescales (bottom left). After hundreds of Alfven timescales, the star settles into the configuration at the bottom right, again with the largest magnetic field vectors at the star’s center.
Figure 4.7: Density plots for neutron stars of magnetic field strength $\|\vec{B}\| = 5 \times 10^{16}$ and $\|\vec{B}\| = 10^{17}$ after evolving for .003s. The disparity in how much these different field strengths deform the star is very apparent.
Figure 4.8: The relative $\beta$ parameters for the field strengths $10^{17} G$ and $5 \times 10^{16} G$. 
Up until this point, we have investigated field configurations that we understand to be unstable in theory. We now look to trigger these instabilities, to investigate their dynamics and search for stable final configurations.

5.1 Poloidal Perturbations

5.1.1 Constant Perturbations

The simplest perturbation that comes to mind is adding a constant rotational velocity to the star. This motion has the potential to wrap the magnetic field lines around the star, adding a toroidal component.

We investigated an initial constant velocity in the $\phi$ direction of $10^5 \text{cm/s}$, which is roughly twice the speed of the Earth’s rotation about its axis. This was time evolved for .1s, or roughly 300 Alfven times. As anticipated, we did see a toroidal component appear within the field almost immediately. Over time, however, it showed no signs of growing, and the magnitude of the poloidal component remains nearly that of the overall field magnitude (figure 5.1). In fact, the toroidal component quickly dies out to less than three orders of magnitude smaller than its poloidal counterpart, as
Figure 5.1: The relative strengths of the average values of the total magnetic field and total poloidal field in an initially poloidal star with rotation. It is apparent that the poloidal component of the magnetic field comprises the overwhelming majority of the field strength.

depicted in figure 5.2.

This is actually an encouraging result. For one, we see the toroidal component comes from a field dragged around as opposed to one that has wound up. The winding is characteristic of differential rotation, so this field evolution indicates consistent rigid rotation. Additionally, these results suggest that a star’s rotation won’t interfere too heavily with its field configuration. This gives us confidence in the applicability of the results from other simulations which neglect the effects of rotation.

5.1.2 Perturbation along neutral line

Wright (1973) and Markey and Taylor (1973) both found that purely poloidal fields could suffer instabilities around the neutral line, or the region where the magnetic field is minimized. With this in mind, we simulated a purely poloidal star with a
velocity perturbation inspired by Lander’s papers [18] and described by:

$$V_{perturb} = (3\cos^2(\theta) - 1)(2\cos(2\phi))/\sin(\theta)\hat{\theta} + \cos(\theta)\sin(\theta)\cos(2\phi)\hat{\phi}$$  \hspace{1cm} (5.1)$$

localized to within ±15% from the neutral line.

Perturbing the star in this manner yields surprising but fascinating results. We finally see the emergence of a consistent toroidal component that is comparable in strength to the poloidal component. We see in figure 5.3 that the two components maintain consistent average strengths that differ by only a factor of three. As expected, this trend becomes apparent after an Alfven time (three cycles in this case).

The rapid introduction of the toroidal component, and the change in field configuration from purely poloidal into a “twisted torus” is apparent in figure 5.4.

Looking specifically at the vector evolution of the toroidal component is interesting as well. The magnitude of the average value oscillates about zero, in what preliminarily appears to be a periodic way (though a longer run time is necessary to
Figure 5.3: The average relative strengths of the magnitudes of the toroidal and poloidal field components in an initially poloidal star with a velocity perturbation. The toroidal component emerges at roughly .003s, after which the magnitude stays fairly constant. The toroidal and poloidal field strengths are less than a factor of 3 apart, which is an encouraging sign for stability.

say conclusively) (figure 5.5). This does not mean the toroidal field is vanishing—rather, it has vectors moving in opposing directions that alternate being strongest in magnitude.

While these results are highly promising, there are some features which suggest that the code isn’t processing the simulation entirely the right way. One such concern arises from looking at evolution of the toroidal field vectors (figure 5.6). We see a vector pile up at the $\theta$ boundary during certain time steps, which defy the constraints of our grid. This is likely due to numerical errors in both VisIT and
5.1.3 $M=2$ Instability

Our arbitrary configuration yielded interesting results, which inspires a deeper investigation into instabilities along the neutral line. As in the above section, we localize the instability to within $\pm 15\%$ of the neutral line, but this time we are more...
FIGURE 5.5: The average value of the toroidal component of the magnetic field oscillates about 0.

deliberate and mathematical.

In poloidal field configurations, the instability in the fluid velocity is expected to be triggered with an $m=2$ azimuthal distribution [18]. To obtain our expression, we begin with the definition of the vector spherical harmonics:

\[ \Phi_{lm} = \mathbf{r} \times \nabla Y_{lm}. \]  \tag{5.2}

Recalling that

\[ Y_2^2 = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\phi} \]  \tag{5.3}
Figure 5.6: Toroidal field vectors. We see the emergence of toroidal vectors, the evolution of the toroidal field, and vector build up at the $\theta$ boundary.

we obtain

$$\Phi_{22} = \sqrt{\frac{15}{8\pi}} \sin\theta \left[ \cos 2\phi \cos \theta \hat{\phi} + \sin 2\phi \hat{\theta} \right]$$

(5.4)

as our velocity perturbation.

The results of this simulation are very similar to the previous one using an arbitrary expression. We see the development of a toroidal component over a very similar timescale, as well as similar overall field structure over the course of the field evolution (which runs for .1s). The behavior of the toroidal field vectors is mirrored as well.

It is encouraging that both perturbations yield similar results, as this indicates that the instabilities and the configuration are a generic feature of the system and
not a consequence of numerical errors particular to an initial condition.

**Figure 5.7**: Magnetic field evolution of an initially poloidal field perturbed with an m=2 instability in velocity.

### 5.1.4 Boundary Constraints

In Chapter 4, we postulated that fixing the field at the boundary to its initial configuration would suppress instabilities in a star’s magnetic field. It then seemed natural to investigate the effects of these constraining boundary conditions on magnetic field development when there is an initial instability.

Fixing the field at the boundary in these particular simulations proved to create regions of extremely low density, which in turn led to the collapse of the star within a few ten-thousandths of a second. This can best be explained by the fact that the boundary in which we are imposing these conditions, the edge of the grid, is not the same as the star boundary. With these instabilities causing rapid initial evolution for the magnetic fields within the star, there may be some issues in the code with keeping the field within the star continuous over the region between the star’s surface and the edge of the box. When decreasing the field strength by a factor of two at the boundary and maintaining the same configuration, the code runs for .022s before
indicating a negative density, which is thousands of times longer than it previously had (though not as long as without the additional boundary constraints). This is highly unexpected, and could potentially be fixed on a grid in which the r boundary is further from the star boundary. The outer boundary is also ‘reflective’ for this simulation, which may cause oscillations. This will be worth investigating in the future.

5.2 Toroidal Perturbations

Markey and Taylor [19][20] also showed that in toroidal configurations there exists an m=1 stability localized around the symmetry axis. Following the same mathematical procedure as for purely poloidal fields with an m=2 instability, we find:

\[
\Phi_{11} = \sqrt{\frac{3}{8\pi}} \left( -\sin \theta - \cos \phi \cos \theta \phi \right)
\]  

(5.5)
\[
\Phi_{21} = \sqrt{\frac{15}{8\pi}} \left( -\sin \phi \cos \theta \dot{\theta} - \cos \phi \cos 2 \theta \dot{\phi} \right). \tag{5.6}
\]

At this time, preliminary results from the perturbed toroidal simulations do not show the development of new field structure. A very small poloidal component develops of magnitude $10^6 \text{G}$, which is negligible next to the strong toroidal field of $10^{16} \text{G}$. An immediate next step is to try perturbing at different parts of the star and explore other perturbations for toroidal fields. Hopefully soon there will be interesting field evolution results.
Gravitational Wave Strain

It is now time to assess the magnetars we have simulated for their potential to generate detectable gravitational radiation. We consider the case of \( l=m=2 \) instability in a poloidal field, as this is our most dynamic and realistic simulation. Earlier, we discussed the methodology for calculating wave strain \( (h) \) when the magnetic and rotational axes are nonaligned, via moment of inertia \( (I) \) and ellipticity \( (\epsilon) \)

\[
\epsilon = \frac{I_{zz} - I_{xx}}{I_0} \quad (6.1)
\]

\[
I_{jk} = \int_V \rho(r)(r^2\delta_{jk} - x_jx_k)dV \quad (6.2)
\]

\[
h = \frac{16\pi^2 G \epsilon I_0 f^2}{c^4 \frac{r}{r}}. \quad (6.3)
\]

Recall that the moment of inertia of a solid sphere is:

\[
I = \frac{2}{5} MR^2. \quad (6.4)
\]

While it takes a rotating star, with the rotational axis offset from the magnetic axis, to produce gravitational radiation in reality, a calculation with the aligned axes,
which is a significantly simpler simulation, is sufficient for our purposes. Thus, this is what is carried out in the prior calculations in this paper. Evidently, we need a modification to the wave strain if the two axes are aligned, as $I_{zz} - I_{xx} = 0$. We can approximate the deformations in a star with aligned axes for both the poloidal and toroidal fields to first order, as follows[17][21]:

Poloidal Field:

$$\epsilon_p = 10^{-10}\left(\frac{R}{10\text{km}}\right)^2\left(\frac{M}{1.4M_\odot}\right)^{-2}\left(\frac{B}{10^{12}\text{G}}\right)^2$$  \hspace{1cm} (6.5a)

Toroidal Field:

$$\epsilon_T = -10^{-12}\left(\frac{R}{10\text{km}}\right)^4\left(\frac{M}{1.4M_\odot}\right)^{-2}\left(\frac{B}{10^{12}\text{G}}\right)^2$$  \hspace{1cm} (6.5b)

We consider the two components separately, as they deform the star in different ways.

Rotation is also neglected in our model. This is a fair approximation, as magnetars spin very slowly. Even in the more rapidly rotating strongly magnetized newborn neutron stars, if we do not have differential rotation the topology of the surface will hardly be changed (though the growth of instabilities will be affected). With rigid rotation we can continue using our approximation.

We can use VisIT to integrate our magnetic field over the star, allowing us to calculate the ellipticity for both cases we have simulated. To do this, we must separately define expressions within VisIT for the magnitude of both the toroidal and poloidal components divided by the overall star volume. Because we are doing integration over a spherical surface, we must multiply both by $r^2 \sin\theta dr d\theta d\phi$. We manually calculate and enter into our expression $dr d\theta d\phi$, by considering our box size and grid size. We have 50 radial points spanning a $1.2 \times 10^6$ radius, and 20 angular points spanning $2\pi$ and $\pi$ degrees in $\theta$ and $\phi$. Thus, our $dV = \frac{1.2 \times 10^6 \times 2\pi \times \pi}{50 \times 20 \times 20} \approx 1,200$.  

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We obtain for our ellipticity:

At \( t = 0 \)s

\[
\epsilon_{\text{poloidal}} = 6.65 \times 10^{-2} \quad (6.6a)
\]
\[
\epsilon_{\text{toroidal}} = 0.00 \quad (6.6b)
\]

At \( t = .1 \)s

\[
\epsilon_{\text{poloidal}} = 1.34 \times 10^{-1} \quad (6.6c)
\]
\[
\epsilon_{\text{toroidal}} = -2.43 \times 10^{-4} \quad (6.6d)
\]

This tells us that the star becomes progressively more stretched as time goes on.

Calculating wave strain is, again, trivial if one knows the period of the star and its distance away. We can use equation 6.3, the equation for wave strain, to investigate the distances at which a magnetar spinning with a set frequency can produce detectable signals. We know that the minimum wavestrain that LIGO can detect varies with frequency (Section 1.2), and is at a minimum \( h \geq 10^{-23} \) at 100 Hz. Considering \( \epsilon \) after 0.1s and the frequency dependent sensitivity of LIGO, with some simple algebraic manipulation we can plot the minimum distance to a magnetar for detection as a function of its frequency (figure 6.1). At 10 Hz (the lower bound of what LIGO can detect), a star with field strength \( 5 \times 10^{16} \) G can be detected at roughly 460 light years from Earth. In the context of the Milky Way, which is 51,000 ly across, this is not very impressive. However, for frequencies of 30 Hz or greater, the wave strain will be adequate if the star is at a distance of less than 55,000 ly and beyond. This allows for exploration of the entirety of the Milky Way Galaxy.
Figure 6.1: The minimum distance to a magnetar as related to its frequency is shown.
Our simulations and calculations of wave strain reveal that newly born, rapidly rotating and strongly magnetised neutron stars are a promising source for gravitational wave detection by LIGO. Our work also sheds light on poloidal field evolution when perturbed by instabilities around the neutral line, an important step towards building a complete picture of these dynamic and mysterious objects.

From here, there is still much work to be done. Adding rotations, investigating different axial separations and considering temperature in the equation of state will all contribute to the accuracy of our model. We might also consider a wider variety of field configurations, and calculate the ellipticity more directly. Finding a perturbation that triggers the toroidal instability could potentially give rise to another stable twisted torus configuration, allowing us to learn even more about field evolution. Additionally, running all simulations for longer timescales would help ensure that a true equilibrium has been reached.

It is important to note that understanding the dynamics of magnetic field evolution in magnetars is an open problem, and any simulations will be an approximation. We neglect features of neutron stars such as superconductivity and an elastic crust,
and do not use an entirely realistic equation of state. At birth, however, the star will be hot enough that not having an elastic crust is a reasonable approximation. Furthermore, if we find a stable configuration it could become frozen in and later be an input for simulations over longer timescales.

We expect that these effects will make a quantitative, but not qualitative difference, so our solution will give us a first understanding of how magnetic fields evolve. The precise field configuration of magnetars is unknown, and this is one of the first simulations that shows an evolution and runs for long enough to find some kind of equilibrium.

More work is needed to completely and accurately model magnetic field evolution. Computationally, this includes a more fundamental understanding of the boundary conditions and higher grid resolution. Physically this means modeling the plasma outside the star and updating the equation of state. It is likely that some of these modifications will affect our model, but overall we have a very good jumping-off point. After all, by providing a starting point, put ourselves in position to move deeper into exciting physics.
Appendix 1: PLUTO Installation

Two things need to be installed before PLUTO is usable: Python and GNUMake. Python is the easier one to install: The most recent version can be downloaded from their website. GNUMake is a bit harder to install. Download the file from their website http://ftp.halifax.rwth-aachen.de/gnu/make/ (mine was called make-4.1.tar.gz) and install it. Find the folder in downloads, unzip it, and look at the INSTALL document for installation instructions.

If while installing you get a permission denied error, as I did, try instead,
$ sudo make install

Now, this actually didn't work for me it said I didn't have something called autoconf.

If you get an error anywhere in the above text and it says you don't have autoconf, do as follows: Autoconf requires two others to work: Perl, and m4. Perl is already installed on macs, but you will need to get m4. Go to ftp://ftp.gnu.org/gnu/m4/ and choose m4-1.4.tar.gz. The installation process should be quite similar to the one for GNUMake, with instructions in another INSTALL file.

Once these steps are complete, autoconf needs to be installed. It can be found on the site http://ftp.gnu.org/gnu/autoconf/ where you should choose autoconf-latest.tar.gz from the bottom, and install it. Autoconf should now be installed. Now just repeat the GNUMake installation steps.

When you downloaded Pluto, you should have gotten a file called pluto-4.1.tar.gz. Unzip it. Pluto is less clear when it comes to installationyou'll have to do some file moving to make it work. The following steps are a condensed version of the README file in the PLUTO folder.

Steps for installing PLUTO:
Open up finder, and go to your downloads folder. Copy the PLUTO folder. It needs to be pasted into your specific Users folder, which is accessed through Macintosh HD.
In the Users folder (within Macintosh HD), one of the icons should be your current user account, probably called your name. Mine has a little house as an icon. Click on whatever account is the one you are using. Then paste in your PLUTO folder. Now, you need to make a working directory, which is easiest to do in your specific Users folder. This will be where all of your PLUTO files get saved. Open up the terminal and navigate to this folder. Then do the following
$export PLUTO_DIR=/Users/name/PLUTO except, obviously, with the name of your user account instead of ”name”. Again, this is the name of the folder that we pasted the PLUTO folder into. It is case sensitive, so be sure to pay attention to capitalization. This command just defines the term PLUTO_DIR so the program knows where all the PLUTO program files are located. Once you’ve done that, you can do $python $PLUTO_DIR/setup.py

Once you hit enter, it should open up PLUTO’s terminal interface. If you get here then PLUTO has successfully been installed!

To obtain our simulations, some alterations had to be made within PLUTO. The square of the sound speed, which is related to the equation of state by \( c_{\text{sound}}^2 = 2P/\rho \), had to be modified in both the eos.c and fluxes.c files for an isothermal equation of state. In globals.h, \( \gamma \) needed to be modified from 5/3 to 2.
Appendix 2: Lane-Emden Solver

#include<stdio.h>
#include <math.h>

int main(){
float x,y,z,h,z1,z2,z3,z4,y1,y2,y3,y4,dx,n;
printf("Give n:\n");
scanf("%f",&n);
h=.0005;
dx=h;
z1=z2=z3=z4=y1=y2=y3=y4=0;
y=1.0;
z=0.0;
x=0.01;
FILE *NPolytrope;
NPolytrope = fopen("NPolytrope.dat", "w");
while(x<=3.14159)
{
    y1 = h*z;
    z1 = h*-1*pow(x,-2)*(2*x*z + pow(x,2)*pow(y,n));
    y2 = h*(z+z1/2);
    z2 = h*-1*pow(x+dx/2,-2)*(2*(x+dx/2)*(z+z1/2) + pow(x+dx/2,2)*pow(y+y1/2,n));
    y3 = h*(z+z2/2);
    z3 = h*-1*pow(x+dx/2,-2)*(2*(x+dx/2)*(z+z2/2) + pow(x+dx/2,2)*pow(y+y2/2,n));
    y4 = h*(z+z3);
    // Write to file
    fprintf(NPolytrope, "%f %f %f %f %f %f
", x, y, z, y1, z1, y2);
    x+=dx;
}
fclose(NPolytrope);
}
z4 = h*-1*pow(x+dx,-2)*(2*(x+dx)*(z+z3) + pow(x+dx,2)*pow(y+y3,n));
y = y+(y1+2*y2+2*y3+y4)/6;
z = z+(z1+2*z2+2*z3+z4)/6;
x = x+dx;

fprintf(NPolytrope,"%f \t %f \t %f \t %f \n",x,y,z,sin(x)/x);
} return(0);
Bibliography


